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Some problems of musical texts

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Abstract. The aim of this article is to find fixed points and regularities in musical texts, set up statistical tests for their comparison and observe their development. The analysis is based on rank-frequency distributions of pitches. The following indicators are described: the *h*-point and its angle, the *a*-indicator, the *H*-point and the *H*-coverage having an affinity to the golden section, and the *A*-ratio. Different curves capturing the trends are proposed. The analysis has been performed on 266 compositions of 12 European composers from Palestrina to Ligeti.

Key words: h-point, a-indicator, H-coverage, A-ratio, rank-frequency distribution, musical texts

1. Introduction

From the general point of view a musical composition is an organized sequence of the musical sounds[2], musical shapes (motives)[3], and musical sections, sentences, parts, movements, etc., just as linguistic texts are sequences of phonemes, syllables, words, phrases, clauses and sentences. However, both the matter of which they are made and the aim of their production, as well as the inventories of units, are different. Any comparison of their inventory sizes is, nevertheless, futile. But whatever the material or functional background of musical sequences, up to a certain level they display repetitions. Sentences in linguistic texts repeat seldom (except for very colloquial ones), and texts[4], linguistic or musical, never.

The units of musical or linguistic texts are not given a priori; they are constructed by us conceptually. In speech, there is only a stream of sounds with tones, stress and intonation, but without blanks, diacritics, or clear sentence ends. But even this stream can be seen differently by a physicist and a linguist. The physicist constructs waves; the linguist constructs linguistic units and segments the text in many ways. In music a staccato sequence differs musically from a legato sequence, but they are equal as sequence. The segmentation of music (metric segmentation in bars, or ametric segmentation) is not given; it results from a certain rhythm

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[2] We have to distinguish: 1. a simple sound = tone, defined by a combination of pitch, duration, timbre (type of instrumental sound or human voice), intensity and articulation (such as legato, arco, pizzicato, staccato, etc.); 2. a complex sound = vertical set of several simple sounds (intervals, accords, clusters), defined by its composition (the set of participating simple sounds) and its configuration (the set of the order relations on this set, i.e. a 5-tuple consisting of the configurations of pitch, duration, intensity, timbre and articulation); 3. a simple sound group = horizontal sequence of simple sounds (tones), defined by melody, rhythm, meter, colour etc., 4. a complex sound group = sequence of (simple and) complex sounds.

[3] A musical shape = motif representing the smallest meaningful semiotic unit of a musical text; while a simple sound is the smallest structural unit of a musical text.

[4] A musical text is an actual configuration of musical elements, such as a composition, folk, popular, jazz etc.; a song; an improvisation etc. There are various representations of musical texts (scores, instructions, graphical representation, oral tradition etc.).

and meter a posteriori. Hence there is no "natural" unit in musical texts produced by humans. In spite of this, in musical sequences one can observe certain regularities which may but need not be conscious. Those which are conscious are used purposefully by the author; just as a text is partitioned into sentences and chapters, a musical text has sections, parts and movements, etc. But some regularities, local or global, are concealed and must be brought to light by formal methods. In general one says that a special segmentation is prolific if it allows us to discover regularities some of which may be laws. Laws cannot be learnt but they are abode by. If a special order decays – as can be seen in the contemporary music – other order replaces it. The task of science is to capture this order, its decay and the emergence of new order. Needless to say, the transition from one order to another is accompanied by deviations, outliers, extremes and a surface chaos which leads to new equilibria.

A sequence of musical events (sounds) has as many properties as we are able to construct conceptually. Some of them are "more objective", e.g. pitch, duration, intensity, timbre, articulation, density (complex sounds); others are latent and can be interpreted emotionally, e.g. sad, uneasy, magnificent etc. Some of the properties can be measured quite easily; some necessitate personal judgements which are not always unique. Here we shall restrict ourselves to a surface property, namely the frequency of individual tones identified by their pitches. This can be performed either with pencil and paper or using a program which does it automatically. For this purpose we have used Reinhard Köhler's computer program QUAMS (= Quantitative Analysis of Musical Structures) created in 1995/1996, providing distributions from MIDI data, which can also order all used tone pitches in the musical text according to type and frequency, i.e. the program is able to establish rank-frequency distributions of pitch values.[5]

The simplest problem is the computation of the rank-frequency distribution of tone pitches and finding the appropriate theoretical distribution. As has been shown (cf. Köhler, Martináková-Rendeková 1995, 1998; Martináková 1997, 1998; Wimmer, Wimmerová 1997, Martináková-Rendeková 2002, 2003, 2004, 2007) the negative hypergeometric distribution is an adequate model, in most cases also in linguistic texts (cf. Popescu et al. 2007). However, it is not known as yet how to interpret the individual parameters even if their motion is known (cf. Martináková 2007)

Here we shall study some other properties of the rank-frequency distribution.

2. The *h*-point and the *a*-indicator

The *h*-point of a rank-frequency distribution, $f = f(r)$, is a fixed point that can be computed in various ways (cf. Popescu 2007; Popescu et al. 2007; Popescu, Altmann 2007). These ways result from its definition proper, that is to find the point $(r, f(r))$ at which $r = f(r)$, i.e. the rank is equal to frequency. As illustrated in Table 1, in Beethoven´s Sonata No. 6 the *h*-point is located at $r = 46 = f(r)$.

[5] We are deeply grateful to Prof. Reinhard Köhler who kindly placed this program at our disposal.

Table 1
Rank-frequency distribution of tone pitches in Beethoven´s Sonata No. 6

r	f(r)	r	f(r)	r	f(r)	r	f(r)
1	404	16	185	31	85	46	46
2	316	17	181	32	83	47	42
3	303	18	167	33	79	48	36
4	302	19	156	34	78	49	31
5	281	20	150	35	77	50	31
6	278	21	146	36	72	51	29
7	275	22	138	37	70	52	15
8	265	23	129	38	69	53	13
9	247	24	127	39	64	54	11
10	227	25	122	40	59	55	6
11	214	26	113	41	57	56	6
12	208	27	110	42	54	57	5
13	200	28	94	43	53	58	3
14	192	29	89	44	53	59	3
15	187	30	87	45	48		

In some cases for all *r* there is no equal *f*(*r*) and one computes it either exactly (by fitting and interpolation) or one takes that *r* whose *absolute* difference to *f*(*r*) is minimum. For example in Beethoven´s Sonata No. 28 we have

rank r	frequency f(r)	r - f(r)
45	63	-18
46	59	-13
47	56	-9
48	52	-4
49	46	3
50	41	9
51	40	11
52	39	13

where the minimal absolute difference is 3 corresponding to $r = h = 49$.

It has been shown in linguistics that the *h*-point depends on the length of the texts according to the relationship $N = ah^2$, as originally proposed by Hirsch (2005) in scientometrics for the citations count. The indicator

$$a = \frac{N}{h^2} \tag{1}$$

has successfully been used in linguistic text analysis (cf. Popescu et al. 2007; Mačutek, Popescu, Altmann 2007) and brought relevant typological results. The same simple power trend can be seen now in musical texts, as illustrated in Figure 1. Thus, if we compute the *a*-indicators for Beethoven´s Sonatas, as shown in Table 2, we obtain a zero trend, as expected. The almost constant values of *a* can also be found in the last column of Table 2 and in Figure 2. The mean of all sonatas is $\bar{a} = 4.35$. A comparison with Skrjabin having $\bar{a} = 2.84$ shows that the differences are considerable and can have their causes. However, tests for differences must be performed (see below).

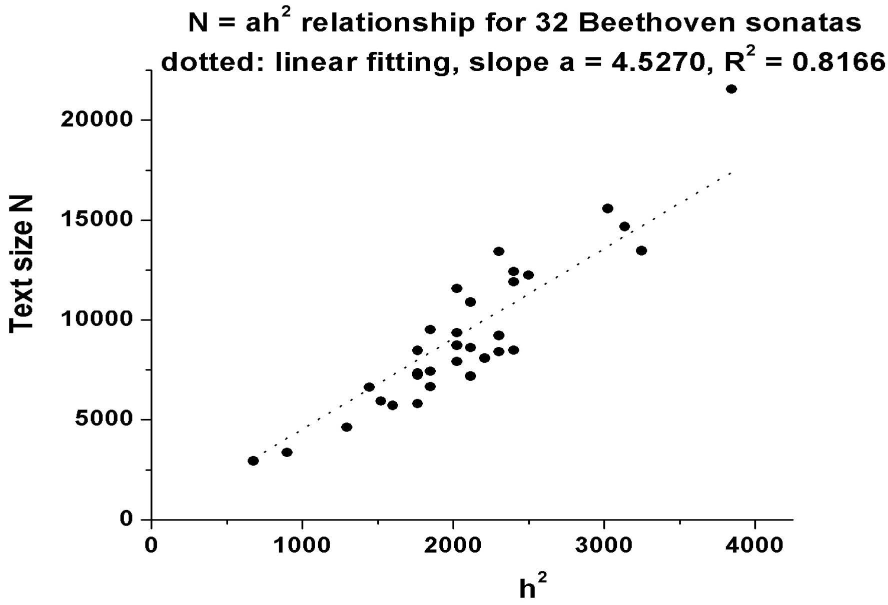

Figure 1. The dependence of *h* on *N* for 32 Beethoven sonatas. Roughly we have $N = ah^2$.

Table 2
The *a*-indicator for Beethoven´s Sonatas

ID	Text	N	h	$a = N/h^2$	ID	Text	N	h	$a = N/h^2$
LvB01	Sonata 1	7332	42	4,16	LvB17	Sonata 17	7905	45	3,9
LvB02	Sonata 2	9340	45	4,61	LvB18	Sonata 18	12428	49	5,18
LvB03	Sonata 3	11915	49	4,96	LvB19	Sonata 19	3362	30	3,74
LvB04	Sonata 4	12248	50	4,9	LvB20	Sonata 20	2937	26	4,34
LvB05	Sonata 5	7229	42	4,1	LvB21	Sonata 21	14682	56	4,68
LvB06	Sonata 6	7171	46	3,39	LvB22	Sonata 22	5802	42	3,29
LvB07	Sonata 7	9201	48	3,99	LvB23	Sonata 23	15575	55	5,15
LvB08	Sonata 8	8396	48	3,64	LvB24	Sonata 24	4619	36	3,56
LvB09	Sonata 9	5706	40	3,57	LvB25	Sonata 25	5930	39	3,9
LvB10	Sonata 10	6623	38	4,59	LvB26	Sonata 26	7416	43	4,01
LvB11	Sonata 11	10898	46	5,15	LvB27	Sonata 27	6643	43	3,59
LvB12	Sonata 12	9497	43	5,14	LvB28	Sonata 28	8467	49	3,53
LvB13	Sonata 13	8461	42	4,8	LvB29	Sonata 29	21559	62	5,61
LvB14	Sonata 14	8597	46	4,06	LvB30	Sonata 30	8713	45	4,3
LvB15	Sonata 15	11581	45	5,72	LvB31	Sonata 31	8075	47	3,66
LvB16	Sonata 16	13439	48	5,83	LvB32	Sonata 32	13468	57	4,15

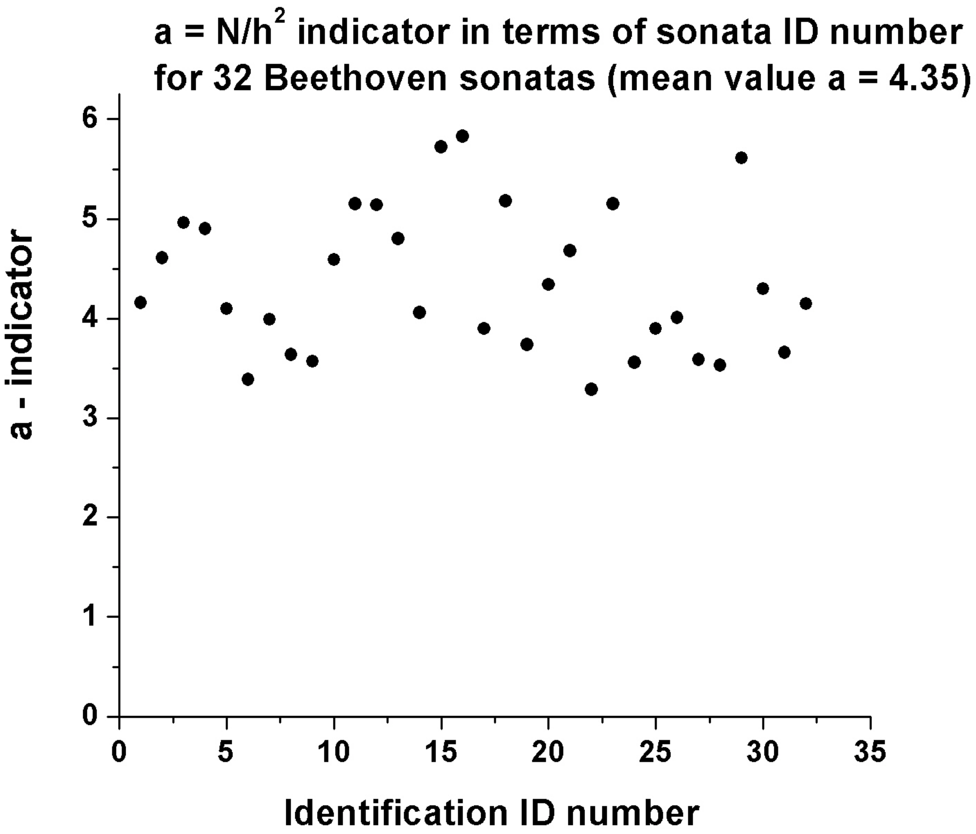

Figure 2. The *a*-values of Beethoven Sonatas

The following hypotheses can be set up in connection with the *a*-indicator: (1) The (mean) indicator *a* is significantly different with different composers either in its mean value or its dispersion. (2) It is significantly different for genres. (3) It may display a certain development tendency in the history of music and it is different for historical musical styles. (4) It is different for compositional language created in different national cultures. Since tests for the *a*-indicators were made possible (cf. Mačutek et al. 2007; Popescu et al. 2008) all hypotheses could be tested. Here we shall restrict ourselves to the comparison of Beethoven and Skrjabin. The basic data of Skrjabin are shown in Table 3, the *a*-indicator is shown in Figure 3.

Table 3
The *a*-indicator for Skrjabin´s compositions

ID	Text	N	h	$a = N/h^2$	ID	Text	N	h	$a = N/h^2$
Skr01	Prelude op. 27 – No 1	355	12	2.47	Skr14	Piece op. 2, No 1	1150	20	2.88
Skr02	Prelude op. 27 – No 2	222	10	2.22	Skr15	Etude op. 8, No 4	747	17	2.58
Skr03	Prelude op. 31 – 1	651	16	2.54	Skr16	Etude op. 8, No 5	1541	21	3.49
Skr04	Prelude op. 31 – 4	155	7	3.16	Skr17	Etude op. 8, No 12	2301	27	3.16
Skr05	Prelude op. 33 – 2	195	8	3.05	Skr18	Poem op. 32 – No 1	981	16	3.83
Skr06	Prelude op. 33 – 3	212	9	2.62	Skr19	Poème tragique op.34	1001	16	3.91
Skr07	Prelude op. 35 – 2	362	11	2.99	Skr20	Etude op. 42, No 4	787	18	2.43
Skr08	Prelude op. 37 – No 1	212	9	2.62	Skr21	Etude op. 42, No 5	3088	32	3.02
Skr09	Prelude op. 37 – No 2	91	5	3.64	Skr22	Sonate No 5, op. 53	7761	50	3.10

Skr10	Prelude op. 48 – 2	224	9	2.77	Skr23	Sonate No 9, op. 68	4014	40	2.51
Skr11	Prelude op. 59	709	17	2.45	Skr24	Poem op. 69 – No 2	539	14	2.75
Skr12	Prelude op. 67 – 1	338	12	2.35	Skr25	Dance op. 73 – No 1: Guirlandes	694	16	2.71
Skr13	Prelude op. 74 – 3	228	11	1.88	Skr26	Dance op. 73 – No 2: Flammes sombres	1051	20	2.63

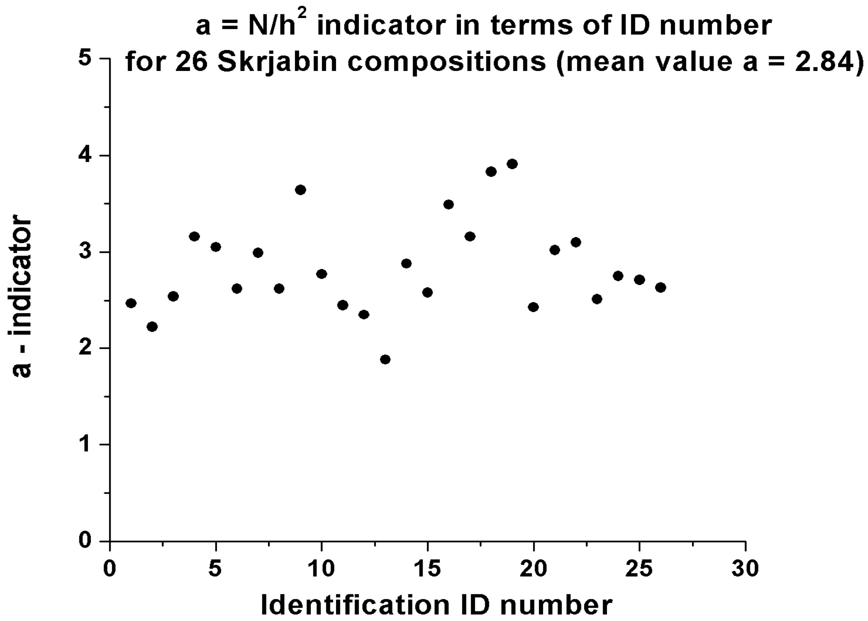

Figure 3. The *a*-indicator in compositions by Skrjabin

The optical difference to Beethoven is evident (Skrjabin´s *a*-indicators are placed deeper than those of Beethoven) but we perform a usual test for averages starting from empirical data. We set up the (simplified) criterion

(2) $$z = \frac{\bar{a}_1 - \bar{a}_2}{\sqrt{Var(\bar{a}_1) + Var(\bar{a}_2)}}$$

which is a standard normal variable (as a matter of fact, with small sample sizes it is a *t*-variable). The individual values can be computed from the above tables mechanically (e.g. by Excel). The variance of *a*-values of Beethoven is $Var(a_1) = 0.50$ and $Var(\bar{a}_1) = 0.50/32 = 0.015625$, that of Skrjabin is $Var(\bar{a}_2) = 0.00889$. Inserting all these values in (2) we obtain

$$z = \frac{4.35 - 2.84}{\sqrt{0.015625 + 0.00889}} = 9.64\ ,$$

telling us that concerning the *a*-indicator and the given compositions, the two composers are very different. The test can be made finer if one estimates a common variance (in that case we would obtain $z = 9.10$).

Nevertheless, the individual composers themselves need not be as homogeneous as they seem when compared with other composers. However, the test between two individual *a*-indicators must be performed in a different way (cf. Mačutek, Popescu, Altmann 2007). The statistics (2) can be used again, but in this case a new problem arises, namely, we do not know the variances of the *a*-indicators. As their theoretical properties are not known, they were estimated from a simulation study. The simulations follow the idea described in Mačutek, Popescu and Altmann (2007), which we recall here in short (Beethoven's Sonata 1 will serve as an example).

Table 4
The *a*-indicator for Palestrina´s Masses

ID	Text	N	h	a = N/h^2	ID	Text	N	h	a = N/h^2
Pls01	Ascendo 1, Motetto	1856	19	5.14	Pls16	Ave Regina Agnus Dei II	402	13	2.38
Pls02	Ascendo 2, Kyrie	898	15	3.99	Pls17	Missa Papae Kyrie	995	16	3.89
Pls03	Ascendo 3, Gloria	1348	17	4.66	Pls18	Missa Papae Gloria	1437	17	4.97
Pls04	Ascendo 4, Credo	2120	19	5.87	Pls19	Missa Papae Credo	2385	19	6.61
Pls05	Ascendo 5, Sanctus	595	14	3.04	Pls20	Missa Papae Sanctus	1060	16	4.14
Pls06	Ascendo 5, Benedictus	563	14	2.87	Pls21	Missa Papae Benedictus	644	13	3.81
Pls07	Ascendo 7, Agnus Dei I	431	13	2.55	Pls22	Missa Papae Agnus Dei I	711	15	3.16
Pls08	Ascendo 8, Agnus Dei II	487	14	2.48	Pls23	Missa Papae Agnus Dei II	793	14	4.05
Pls09	Ave Regina Chant	137	6	3.81	Pls24	Missa Veni Kyrie	669	14	3.41
Pls10	Ave Regina Kyrie	687	15	3.05	Pls25	Missa Veni Gloria	1013	15	4.5
Pls11	Ave Regina Gloria	1357	17	4.7	Pls26	Missa Veni Credo	1596	19	4.42
Pls12	Ave Regina Credo	2355	19	6.52	Pls27	Missa Veni Sanctus	722	14	3.68
Pls13	Ave Regina Sanctus	436	13	2.58	Pls28	Missa Veni Benedictus	576	14	2.94
Pls14	Ave Regina Benedictus	505	13	2.99	Pls29	Missa Veni Agnus Dei I	343	13	2.03
Pls15	Ave Regina Agnus Dei I	396	13	2.34	Pls30	Missa Veni Agnus Dei II	415	14	2.12

We generated 7332 (there are 7332 tones in the sonata) random numbers from the negative hypergeometric distribution[6] with the parameters K = 3.4690, M = 0.8257, n = 59 (parameter values for which the best fit is obtained), and we found the *h*-point and *a*-indicator in this sample. The random number generation is repeated 100 times, resulting in 100 *a*-indicators from samples with the same size and distribution as tone pitches frequencies in Beethoven's Sonata 1. Next, we compute the variance of the 100 *a*-indicators. The procedure is repeated 10 times, i.e., we have 10 variance values, each of them being a variance of 100 *a*-indicators. Their mean is an estimation of the *a*-indicator variance.

We recommend larger number of random samples for a historical or comparative study; here we mainly aim at the method introduction.[7]

Nine compositions were chosen for testing differences between *a*-indicators. Recall that the difference is significant if the *z*-statistics value is less than –1.96 or more than 1.96. The results are shown in Table 5.

[6] The adequateness of the negative hypergeometric distribution for the rank-frequency distribution of tone pitches has been shown in Martináková-Rendeková (2005)
[7] Short simulation programs (cf. also Section 4) written in *R* can be sent upon request (jmacutek@yahoo.com).

Table 5
Tests for differences between *a*-indicators (Beethoven, Palestrina, Skrjabin)

	LvB01	LvB18	LvB31	Pls01	Pls19	Pls29	Skr01	Skr13	Skr19
LvB01	0	**-2.24**	1.18	-1.63	**-3.65**	**2.64**	**2.05**	1.27	0.28
LvB18	**2.24**	0	**3.53**	0.07	**-2.12**	**3.90**	**3.28**	1.84	1.40
LvB31	-1.18	**-3.53**	0	**-2.54**	**-4.50**	**2.06**	1.47	1.00	-0.28
Pls01	1.63	-0.07	**2.54**	0	-1.88	**3.46**	**2.92**	1.77	1.24
Pls19	**3.65**	**2.12**	**4.50**	1.88	0	**4.83**	**4.31**	**2.54**	**2.61**
Pls29	**-2.64**	**-3.90**	**-2.06**	**-3.46**	**-4.83**	0	-0.42	0.08	-1.67
Skr01	**-2.05**	**-3.28**	-1.47	**-2.92**	**-4.31**	0.42	0	0.31	-1.27
Skr13	-1.27	-1.84	-1.00	-1.77	**-2.54**	-0.08	-0.31	0	-1.04
Skr19	-0.28	-1.40	0.28	-1.24	**-2.61**	1.67	1.27	1.04	0

Consider now the dispersion of the *a*-values. Using the unbiased estimators of the variance, we obtain for Skrjabin $Var(a) = 0.2403$, for Beethoven $Var(a) = 0.5197$, but for Palestrina $Var(a) = 1.5249$ though his mean is $a = 3.76$, i.e. it is positioned between Skrjabin and Beethoven, as can be seen in Table 6. Automatically the hypothesis arises whether the dispersion of the *a*-indicators displays a historical development.

To this end we compare the work of some other composers as shown in Table 6.

Table 6
Mean and unbiased variance of *a* of all composers

Name	Mean year	Mean *a*	Variance of *a*
Palestrina (1525-1594)	1560	3.76	1.5249
Gesualdo (1560?-1613)	1587	2.73	0.1810
Monteverdi (1567-1643)	1605	4.60	1.1942
Bach (1685-1750)	1718	3.35	0.2013
Mozart (1756-1791)	1774	5.74	1.0534
Beethoven (1770-1827)	1799	4.35	0.5197
Liszt (1811-1886)	1849	3.01	0.2173
Skrjabin (1872-1915)	1894	2.84	0.2403
Schoenberg (1874-1951)	1913	2.97	0.9905
Stravinsky (1882-1971)	1927	3.56	1.5824
Shostakovich (1906-1975)	1940	2.97	0.7273
Ligeti (1923-2006)	1965	2.20	0.1583

Observing the values of *a* as shown in Figure 4, we can see that the existing trend is clearly divided in two parts: the first from Palestrina up to Mozart, the second from Mozart down to Ligeti. The first part cannot be captured by any simple curve but the second part displays a monotone linear decreasing trend ($R^2 = 0.73$) as can be seen in Table 7, yielding $a = 29.4730 – 0.0138t$, where t is the given mean year.

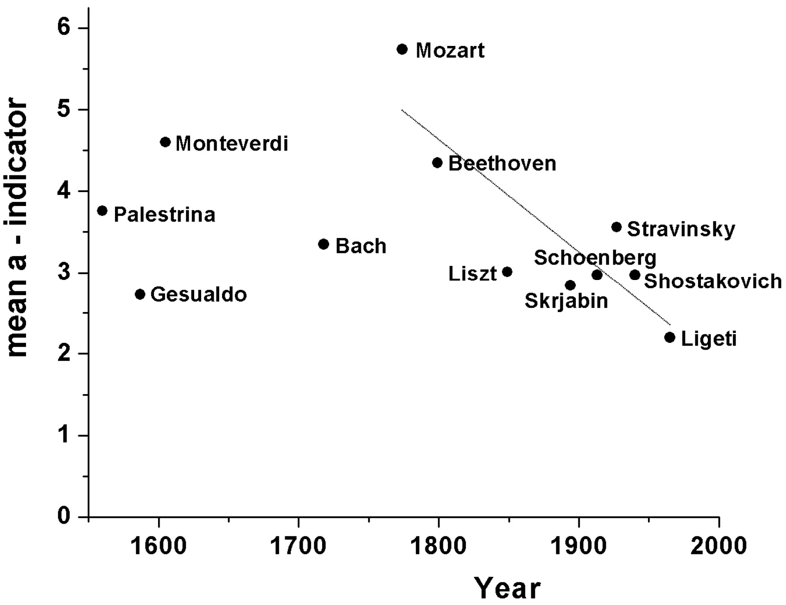

Figure 4. The trend of *a*-values

Table 7
The *a*-trend beginning with Mozart

Year	*a*-observed	*a*-computed
1774	5.74	4.99
1799	4.35	4.65
1849	3.01	3.96
1894	2.84	3.34
1913	2.97	3.07
1927	3.56	2.88
1940	2.97	2.70
1965	2.20	2.36

We conjecture that the complete trend is curvilinear and concave but the *a*-indicator should be computed from the complete work of each composer. This is unfortunately a very tiresome task that can be performed only partially in the future.

3. The view angles

In linguistic texts the *h*-point is considered a control position: the writer subconsciously looks at the top and the end of the distribution (the top is represented by f_1 – the greatest frequency, the end by the text vocabulary *V*) and controls their development. The angle of the *h*-point is metaphorically called "writer´s view". But the situation is quite different in music. The tone pitches are not parallels of words but rather of phonemes or letters. The composer cannot develop any other tones than those given by the instruments, but a speaker develops words continuously. Hence the LNRE (large number of rare events) theory does not hold for this aspect of music. Nevertheless, it can be shown that the rank-frequency of pitches abides by the negative hypergeometric distribution, which is used also in modelling the rank-frequency of letters or phonemes. A further difference is the fact that the angle of "writer´s view" in linguistic texts converges to the golden section 1.618. (cf. Popescu, Altmann 2007) but phonemes/letters or tone pitches do not. Nevertheless, the angle can be characteristic of composition, author, style, genre, historical epoch, etc., just as it is with other properties of rank-frequency distributions (cf. Martináková 2007).

Consider the *h*-point and the cosine of its angle as presented in Figure 5. The cosine can be computed as

$$\cos\ \alpha = \frac{-[h(f_1 - h) + h(n - h)]}{[h^2 + (f_1 - h)^2]^{1/2}[h^2 + (n - h)^2]^{1/2}}$$

where f_1 is the greatest frequency and n is the inventory of pitches. For example Sonata 1 by Beethoven in which $h = 42$, $f_1 = 537$, $n = 59$ yields

$$\cos \alpha = -[42(537-42) + 42(59-42)]/\{[42^2 + (537-42)^2]^{1/2}[42^2+(59-42)^2]^{1/2}\} = -0.9553$$

from which arccos(-0.9553) = 2.8416 radians. Evidently, these values drastically differ from those in linguistic texts concerning words which converge to the golden section.

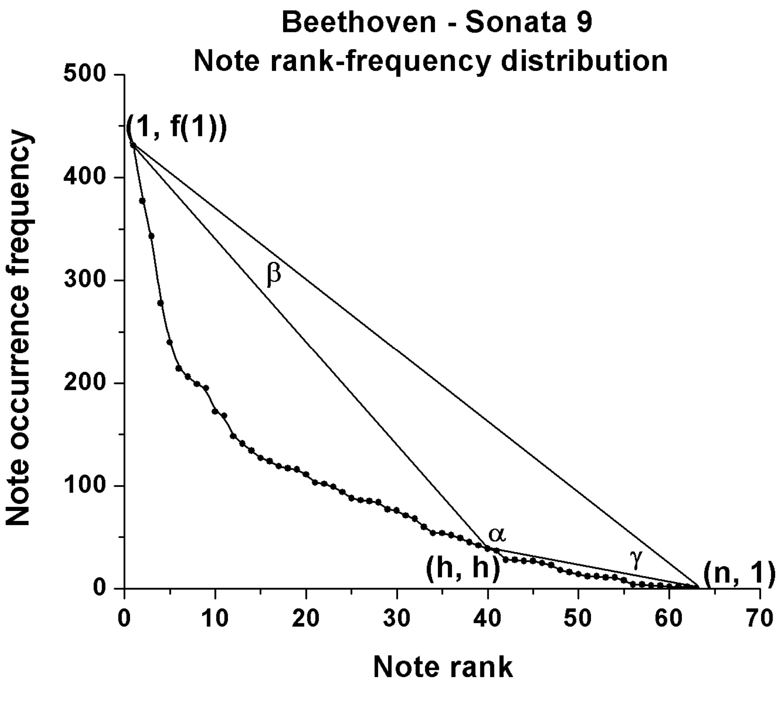

Figure 5a. The *h*-point and the angle α for a Beethoven composition

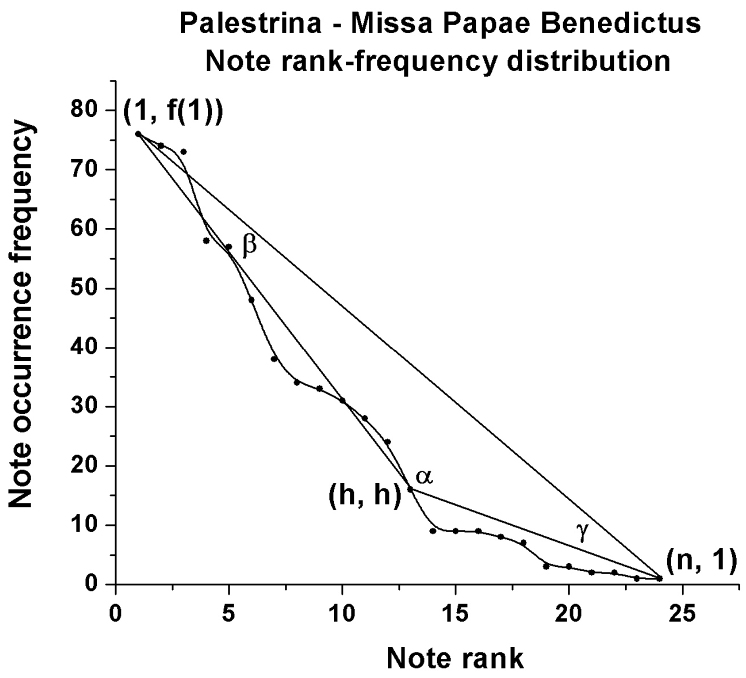

Figure 5b. The *h*-point and the angle *α* for a Palestrina composition

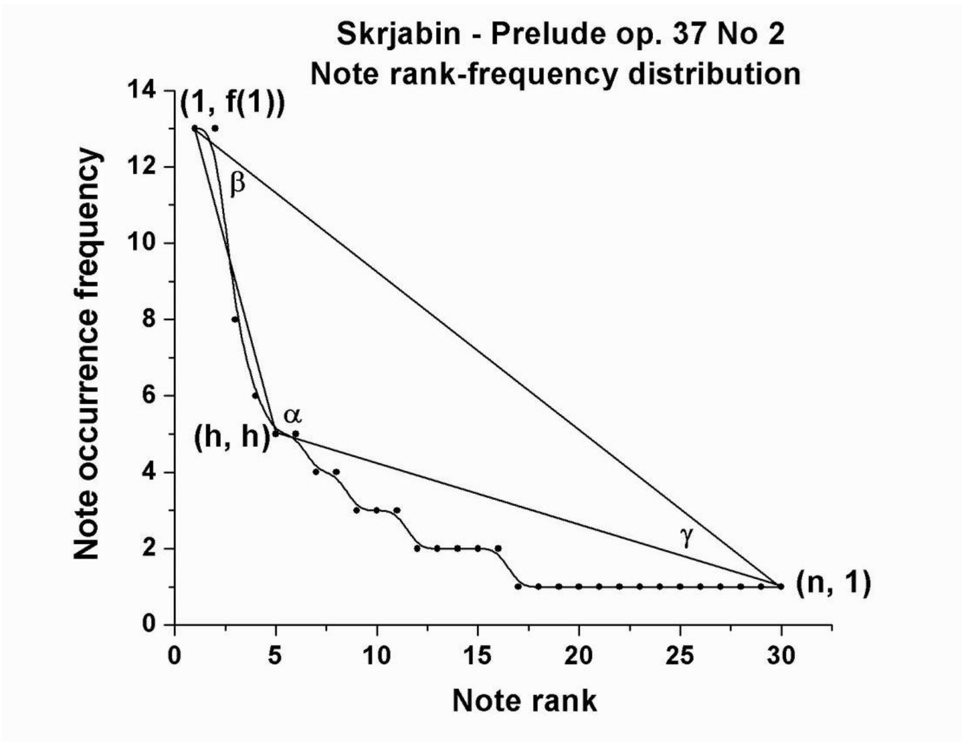

Figure 5c. The *h*-point and the angle α for a Skrjabin composition

As can be seen in Figure 6, the angles with Palestrina do not depend on composition length, and the angles *β* and *γ* are so acute that they cannot be used for characterization.

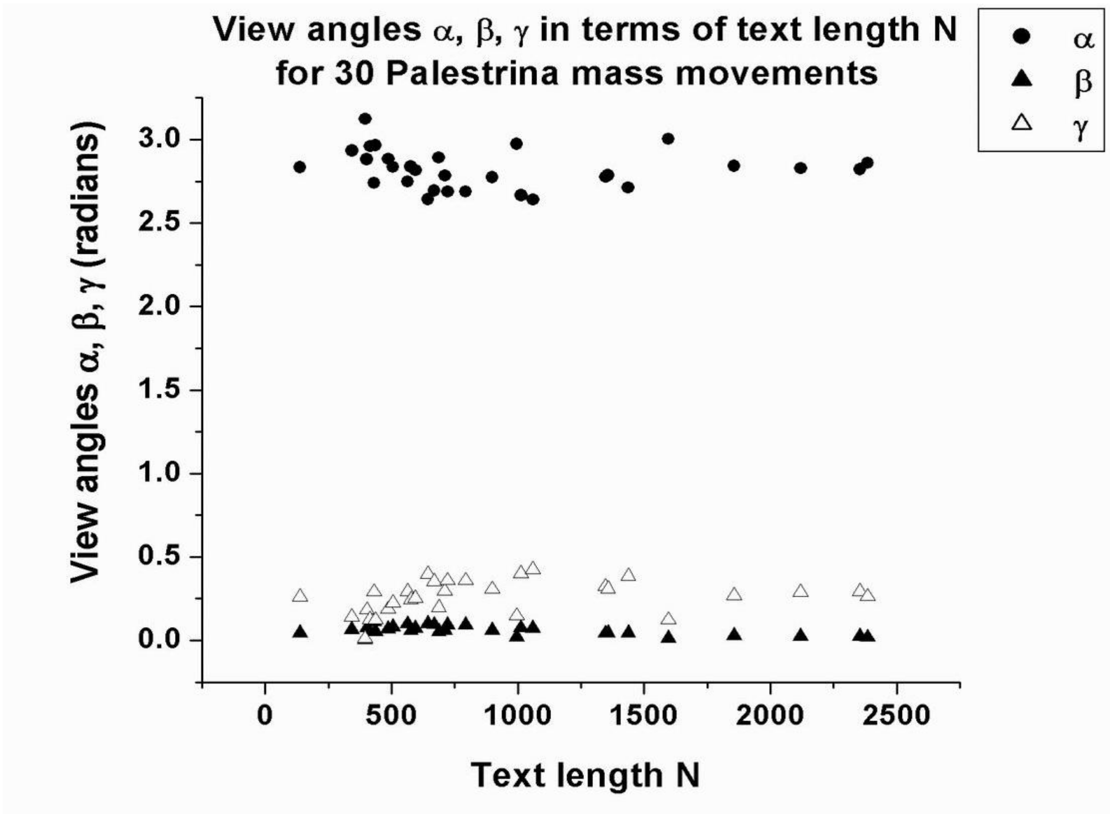

Figure 6. The angles of the triangle of the pitch distribution with Palestrina

Again, we can look whether there is a development of this angle in time. In Table 8 one can see the mean α radians with different composers

Table 8
The alpha radians with different composers

Name	Mean α radians
Palestrina	2.8212
Gesualdo	2.6053
Monteverdi	2.6945
Bach	2.6510
Mozart	2.6929
Beethoven	2.8340
Liszt	2.5526
Skrjabin	2.5582
Schoenberg	2.5449
Stravinsky	2.5615
Shostakovich	2.5515
Ligeti	2.8005

The mean α radians seem to represent a constant which does not change in the course of time and displays only a random oscillation. Hence this indicator is evidently a musical constant having a value $\alpha = 2.6557 \pm 0.1071$, almost coincident with the mathematical (Euler's or Napier's) number $e = 2.71828...$

3. Searching for the golden section

In natural language texts the golden section has been found as the limit of the α radians of the *h*-point of the rank-frequency distribution of words. However, as mentioned above, in music, simple notes do not correspond to words in language but rather to phonemes or letters. Hence if we believe in the existence of the golden section in the distribution of pitches, we must search for it differently. Let us begin with presenting the ranks and the frequencies in logarithmic form as can be seen in Table 9 for Sonata 5 by Beethoven. The natural logarithm of the rank is in the third column, the logarithm of the frequency in the fourth. If we draw a diagram, the logarithmic presentation has approximately the form of a concave monotone decreasing function, as illustrated in Figure 7.

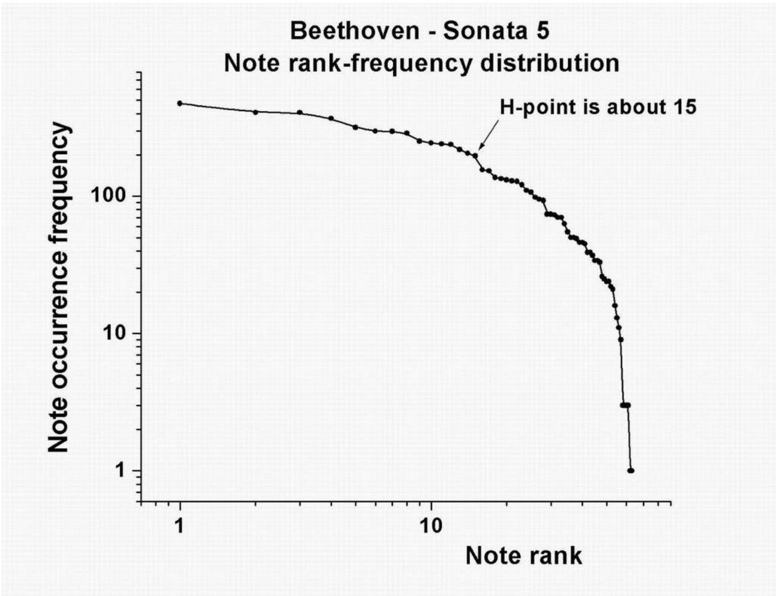

Figure 7. *h*-point definition

However, one can see that the first part of this curve has a rather linear form. Let us seek the end of the straight line. To this end we first take the first three values (of *log(r)* and *log(f(r))*) and compute the straight line. We obtain *log(f(r))* = 6.1457 – 0.1454 *log*® and the determination coefficient is R^2 = 0.8668. We add the next value and compute the straight line again. In this way we continue up to r = 18. The straight line exists if the determination coefficient R^2 oscillates or even increases, as can be seen in the sixth column of Table 9. Beginning with point r = 15 the determination coefficient begins to decrease because the points change the direction. Hence point r = 15 is the last point of the straight line.

Now let us compute the cumulative relative frequencies of the first part of the rank-frequency distribution as shown in the seventh column of Table 9. As can be seen, $F(15)$ = 0.6159 represents that value which is the nearest to the golden proportion 0.618. This *r*-point will be called H and the cumulative frequency $F(H)$ is called H-coverage.

Table 9
Computation of the *H*-point (Beethoven Sonata No 5)

Rank r	Frequency f(r)	ln(r)	ln(f(r))	ln(f(r)) = a-b ln(r)	R^2	F(r)
1	473	0.0000	6.1591			0.0654
2	407	0.6931	6.0088			0.1217
3	407	1.0986	6.0088	6.1457-0.1454x	0.8668	0.1780
4	369	1.3863	5.9108	6.1519-0.1636x	0.9213	0.2291
5	317	1.6094	5.7589	6.1760-0.2159x	0.8675	0.2729
6	298	1.7918	5.6971	6.1927-0.2451x	0.8876	0.3142
7	296	1.9459	5.6904	6.1970-0.2517x	0.9123	0.3551
8	288	2.0794	5.6630	6.1988-0.2540x	0.9285	0.3949
9	252	2.1972	5.5294	6.2161-0.2748x	0.9221	0.4298
10	244	2.3026	5.4972	6.2288-0.2889x	0.9263	0.4635
11	240	2.3979	5.4806	6.2365-0.2970x	0.9341	0.4967
12	239	2.4849	5.4765	6.2395-0.2999x	0.9418	0.5298
13	219	2.5649	5.3891	6.2499-0.3095x	0.9434	0.5601
14	206	2.6391	5.3279	6.2628-0.3208x	0.9416	0.5886
15	197	2.7081	5.2832	6.2758-0.3318x	0.9400	0.6159
16	155	2.7726	5.0434	6.3111-0.3605x	0.8939	0.6373
17	153	2.8332	5.0304	6.3394-0.3825x	0.8798	0.6585
18	137	2.8904	4.9200	6.3724-0.4074x	0.8627	0.6774
……	….	….	….	….	….	….

Of course, this lengthy computation is not always necessary because *H* can be determined visually or using a very quick method by means of Excel. The *H*-point is given by the rank at which *r***f*(*r*) becomes a maximum, as shown in Table 10 for the same data and in Figure 8.

Table 10
Computing the *H*-point (Beethoven Sonata No. 5)

Rank *r*	Frequency *f*(*r*)	*r***f*(*r*)
1	473	473
2	407	814
3	407	1221
4	369	1476
5	317	1585
6	298	1788
7	296	2072
8	288	2304
9	252	2268
10	244	2440
11	240	2640
12	239	2868

13	219	2847
14	206	2884
15	197	2955
16	155	2480
17	153	2601
18	137	2466
19	134	2546
20	131	2620
…………	…………………	………………

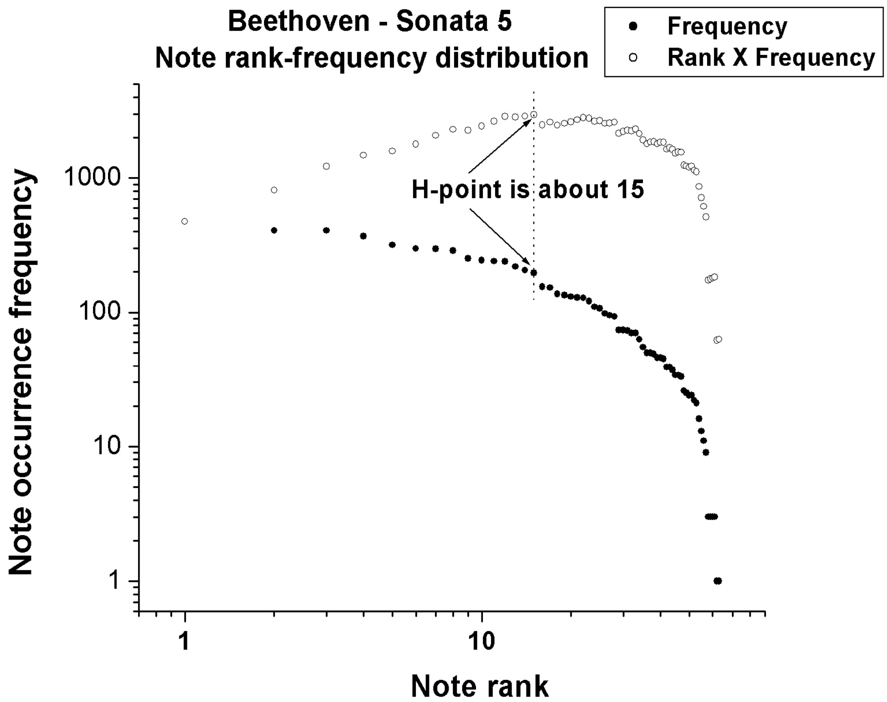

Figure 8. Determination of the *H*-point

F(*H*) is not always exactly 0.618 but it tends to this number. We must take into account that in written compositions the composers can make changes in the score a posteriori and cause thereby deviations, while in improvisations the agreement could be almost exact. To this end examinations in this direction should be made.

In order to show that this point displays a certain stability and is part of the composition we show in Figure 9 the *F*(*H*)-coverage for all Sonatas of Beethoven. The coverage does not change either with the length of the composition or with Beethoven´s age, and its mean for all Sonatas is 0.617 ± 0.057 where 0.057 is the standard deviation σ (see Table 10). Possibly the partitioning of the Sonatas in their parts would bring still better agreement.

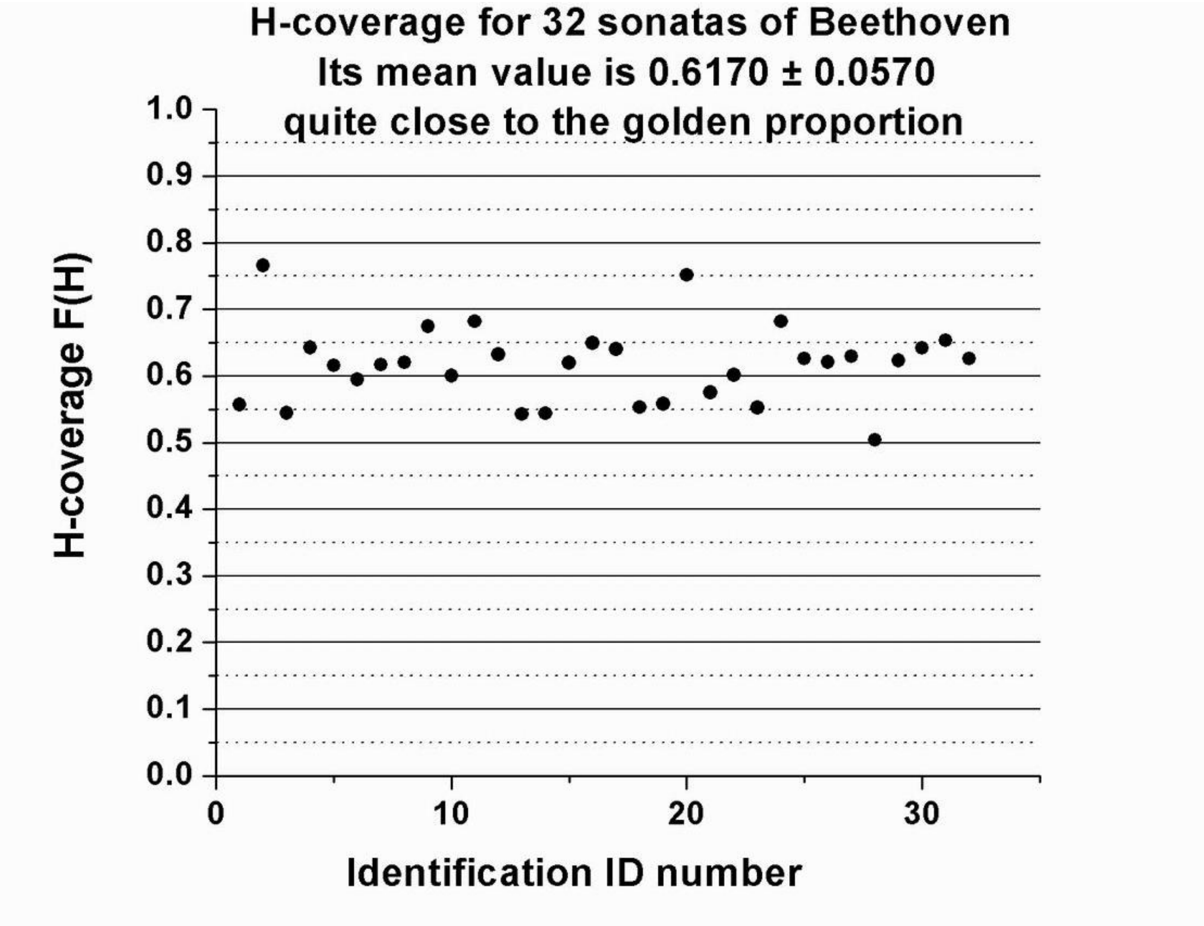

Figure 9. The $F(H)$ for Beethoven´s Sonatas

In (linguistic) text analysis one knows that the most frequent words are synsemantics but in music we must look for the function of these pitches. Let us start from the usual marking of tones as shown in Figure 10, where the middle c is at piano keyboard ($c^1 = 60$).

Octaves	**Note Numbers**											
	C	**C#**	**D**	**D#**	**E**	**F**	**F#**	**G**	**G#**	**A**	**A#**	**B**
$C_3 – B_3$	0	1	2	3	4	5	6	7	8	9	10	11
$C_2 – B_2$ Sub-Contra Octave	12	13	14	15	16	17	18	19	20	21	22	23
$C_1 – B_1$ Contra Octave	24	25	26	27	28	29	30	31	32	33	34	35
C – B Great Octave	36	37	38	39	40	41	42	43	44	45	46	47
c – b Small Octave	48	49	50	51	52	53	54	55	56	57	58	59
$c^1 – b^1$ One-Line Octave	60	61	62	63	64	65	66	67	68	69	70	71
$C^2 – b^2$ Two-Line Octave	72	73	74	75	76	77	78	79	80	81	82	83

$c^3 – b^3$ Three- Line Octave	84	85	86	87	88	89	90	91	92	93	94	95
$c^4 – b^4$ Four-Line Octave	96	97	98	99	100	101	102	103	104	105	106	107
$c^5 – b^5$ Five-Line Octave	108	109	110	111	112	113	114	115	116	117	118	119
$c^6 – b^6$	120	121	122	123	124	125	126	127				

Figure 10. MIDI note numbers for the tone pitch and octave designation according to the Helmholtz System used in this article

The musicological interpretation of the points *H* and *h* could be, for example, the *Sonata No. 5* by Beethoven shown in Table 11, as follows: the Sonata is composed in tonal system in *C minor key* (1. movement – Allegro molto e con brio), in *A-flat major key* (2. movement – Adagio molto), *C minor key* (3. movement – Finale, last chord is in *C– major*, similar as in modal system where the last chord is mostly in major version: last cadence: minor subdominant triad: *f-ab-c*; diminished seventh (vii7 in minor keys): *h-d-f-ab* and tonic in major version: *c-e-g*.

In the first most frequent 15 tones (to the point *H*) we can see only the basic tones of the C minor key: *c-d-eb-f-g-ab-bb* in natural version (cf. Aeolian modus).

The tones from the point *H* to *h* (15-42) are:

1. the same tones but placed also in other octaves;
2. one most frequent new tone: *b* – it is very important as major seventh which is the basic tone (mediant) in the dominant (*g-b-d*);
3. one less frequent tone: *d-flat* – it is the basic tone in *A-flat major key* in the second movement;
4. two diesis: *e*, *a* – depend on the leading tones in melody and chromatization (*e* is also the mediant in major version of tonic triad *c-e-g* and *a* is the mediant for subdominant triad *f-a-c*);

After the point **h** we can find except for the mentioned tones (but also in more extreme octaves) the last 12th tone *f#/g-flat*.

Table 11
Pitches corresponding to ranks and frequencies in Beethoven´s Sonata 5

Rank	Freq	Pitch	Name	Rank	Freq	Pitch	Name	Rank	Freq	Pitch	Name
1	473	6300	e-flat^1	22	128	7100	b^1	43	39	3400	B-flat/A#$_1$
2	407	6000	c	23	121	5000	d	44	37	5400	g-flat/f#
3	407	5500	g	24	110	4600	B-flat/A#	45	34	4500	B-flat/A#
4	369	5100	e-flat	25	107	6100	d-flat/c#^{1}	46	34	3100	G$_1$
5	317	6700	g^1	26	98	8000	a-flat^2	47	33	6600	g-flat/f#^{1}
6	298	5600	a-flat	27	95	4400	A-flat	48	26	7800	g-flat/f#^{2}
7	296	7200	c^2	28	93	8200	b-flat/a#^{2}	49	25	3200	A-flat$_1$
8	288	5800	b-flat/a#	29	74	7300	d-flat/c#^{2}	50	24	8100	a^2
9	252	5300	f	30	74	6400	e^1	51	24	4200	G-flat/F#
10	244	6500	f^1	31	73	3900	E-flat/D#	52	22	8500	d-flat/c#^{3}
11	240	7500	e-flat^2	32	70	8600	d^3	53	21	4900	d-flat/c#

12	239	6800	a-flat^1	33	70	4700	B	54	16	3500	B$_1$
13	219	4800	c	34	63	8700	e-flat^3	55	13	3800	D
14	206	6200	d^1	35	55	8300	b^2	56	11	8800	e^3
15	197	7000	b-flat/a#^{1}	36	50	6900	a^1	57	9	3300	A$_1$
16	155	7400	d^2	37	50	5200	e	58	3	4000	E
17	153	7700	f^2	38	49	8900	f^3	59	3	2900	F$_1$
18	137	7900	g^2	39	46	3600	C	60	3	3000	G-flat/F#$_1$
19	134	8400	c^3	40	46	4100	F	61	3	3700	D-flat/C#
20	131	5900	b	41	45	7600	e^2	62	1	9000	g-flat/f#^{3}
21	129	4300	G	42	39	5700	a	63	1	2700	E-flat/D#$_1$

The computation of H and $F(H)$ is shown in Tables 1A to 12A in the Appendix. As can be seen in Tables 1A to 12A and presented collectively in Table 12, the mean $F(H)$ seems to develop. With Palestrina it does not acquire its ideal form; with Bach it acquires its purest form, thereafter an oscillation begins. This statement is very preliminary because we studied only some works by several composers. A more extensive investigation is necessary in order to attain better founded statements. In any case we have shown that something like the golden proportion exists directly in the frequencies of pitches.

Table 12
Survey of *H*-coverages

Composer	**mean F(H)**	**σ**
Palestrina	0.7530	0.0893
Gesualdo	0.6160	0.0249
Monteverdi	0.6183	0.0972
Bach	0.6180	0.0703
Mozart	0.6076	0.0530
Beethoven	0.6170	0.0570
Liszt	0.6231	0.0692
Skrjabin	0.5766	0.0813
Schoenberg	0.6268	0.0208
Stravinsky	0.7556	0.1079
Shostakovich	0.6746	0.0886
Ligeti	0.6986	0.0491

Since the computation of H is not always unequivocal but we are aware of its existence, the following algorithm can be proposed a posteriori: (a) Plot the ranks and frequencies of pitches in double-logarithmic scale. (b) Determine the H-point optically as the last point on the straight line beginning with $\ln(f_1)$. (c) Compute stepwise the linear regression starting from the point $<0, \ln(f_1)>$ down to the point yielding the last maximum determination coefficient. (d) If the optical and the computed H-point coincide, accept it. (e) If they do not coincide, choose that of the two points whose $F(H)$ is nearer to 0.618. (f) Check the computation by the rank H corresponding to $\max[r{*}f(r)]$. (g) Generally, a major downwards bend of the actual distribution defines the H-point, as illustrated in Figure 11. This implies that it is located at the maximum of the difference $\Delta f = f_{actual} - f_{fitting}$, as shown in Figure 12. It is to be noticed, however, that the parasite maxima at lower ranks should be discarded. Moreover, this last method should be applied cautiously, inasmuch as irregular actual distributions may produce a few Δf maxima before the occurrence of the major distribution bend.

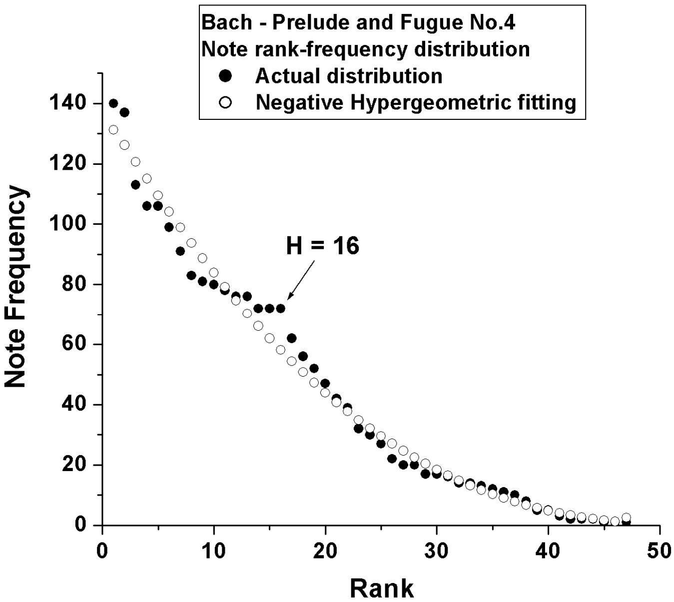

Figure 11. The *H*-point as a distribution break up

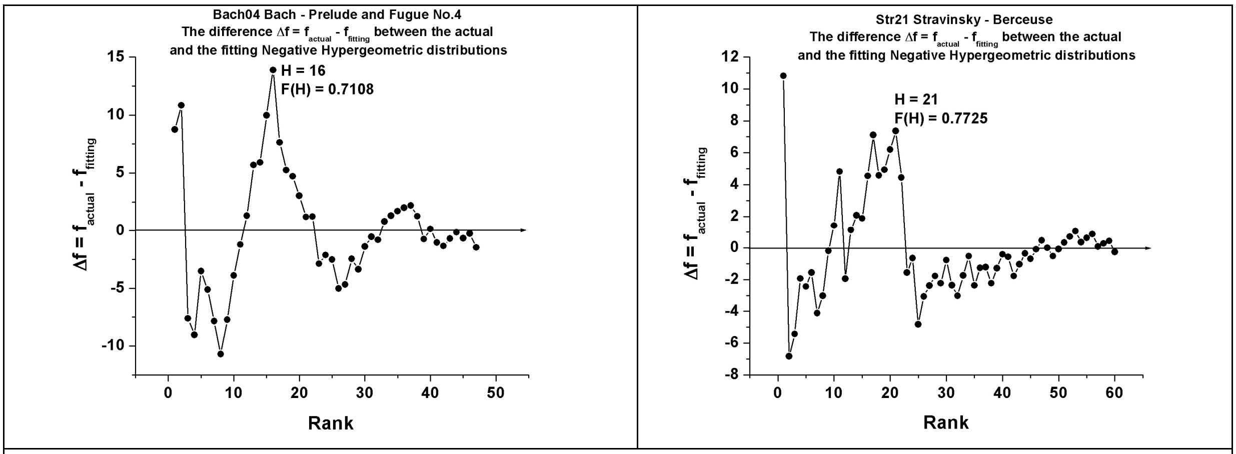

Figure 12. The *H*-point as the maximum of the $\Delta f = f_{actual} - f_{fitting}$ difference

The differences between $F(H)$ coverages can again be tested using formula (2). Variances were estimated from simulations (cf. Section 2). Again, nine compositions (by Beethoven, Palestrina and Skrjabin) were chosen.

Table 13
Tests for differences between some $F(H)$ coverages

	LvB01	LvB02	LvB28	Pls01	Pls15	Pls23	Skr01	Skr07	Skr14
LvB01	0	**-2.28**	0.56	**-2.74**	0.09	**-3.02**	0.69	0.80	**-1.78**
LvB02	**2.28**	0	**2.88**	-0.79	1.78	-1.24	**2.40**	**2.54**	0.08
LvB28	-0.56	**-2.88**	0	**-3.26**	-0.33	**-3.49**	0.27	0.37	**-2.26**
Pls01	**2.74**	0.79	**3.26**	0	**2.24**	-0.47	**2.81**	**2.95**	0.74
Pls15	-0.09	-1.78	0.33	**-2.24**	0	**-2.54**	0.50	0.58	-1.51

Pls23	**3.02**	1.24	**3.49**	0.47	**2.54**	0	**3.07**	**3.20**	1.14
Skr01	-0.69	**-2.40**	-0.27	**-2.81**	-0.50	**-3.07**	0	0.08	**-2.06**
Skr07	-0.80	**-2.54**	-0.37	**-2.95**	-0.58	**-3.20**	-0.08	0	**-2.17**
Skr14	1.78	-0.08	**2.26**	-0.74	1.51	-1.14	**2.06**	**2.17**	0

As can be seen, significant differences can arise even within the work of one composer and about half of the differences are significant. Hence $F(H)$ seems to be a very sensitive characteristic of the composition.

Consequently, the question arises whether $F(H)$ is a historically changing phenomenon or simply a text characteristic. Its "ideal value" attained by Bach displays a motion beginning with Palestrina and ending (preliminarily) with Ligeti, but this motion is not very smooth. In any case one can see a concave course. A special representation of this trend is shown in Figure 13, where we plotted the dependence $\langle$time, Log $A\rangle$ with $A = 1/\left|[F(H) - 0.618034]\right|$ as a merit indicator. Clearly we have to deal with the time development of a couple of concurring processes, firstly a fast rising one and secondly a slowly decaying one. Most intuitive appears the comparison of this compound motion in terms of the difference of two exponential functions as follows

$$y(t) = c\left[\exp\left(-\frac{t-t_0}{T_{fall}}\right) - \exp\left(-\frac{t-t_0}{T_{rise}}\right)\right]$$

where y is the considered musical merit indicator (here Log A), t is the time, t_0 is the time origin, c is a scaling factor, T_{rise} is the rise time of the "musical phenomenon", and T_{fall} is its decay time. This is a slightly modified 4 parameter Box-Lucas2 fitting exponential function built in the Origin 6.1 program (see more in Box, Lucas 1959). As it is illustrated in Figure 13, the musical golden proportion impetus has a maximum located in the mid of the 17th century, a rise time $T_{rise} \approx 75$ years, and a decay time $T_{fall} \approx 150$ years, hence a width of about $W = T_{rise} + T_{fall} = (75 + 150)$ years = 225 years, heralding and covering the brilliant epoch of Bach, Mozart, and Beethoven. On the other hand, the oldest composers considered in the present paper and belonging to the beginning of this motion are Palestrina, Gesualdo and Monteverdi after Leonardo da Vinci (1452–1519), Michelangelo (1475–1564), and Luca Pacioli (1445–1514) with his *Divina Proportione* (1509). Consequently, it appears that the whole musical golden proportion inspiration appears as a late echo of the Renaissance that spans roughly the 14th through the 17th century.

This development can be seen in Table 14 and Figure 13.

Table 14
Fitting A =1/|meanF(H)-0.618034| by Box-Lucas and impulse functions

Composer	Year	mean F(H)	A	Log A	$(\text{Log } A)_{\text{Box-Lucas}}$	$(\text{Log } A)_{\text{impulse}}$
Palestrina	1560	0.7530	7.409	0.870	0.802	0.797
Gesualdo	1587	0.6160	491.642	2.692	2.817	2.821
Monteverdi	1605	0.6183	3759.398	3.575	3.594	3.599
Bach	1718	0.6180	29411.765	4.469	3.848	3.842
Mozart	1774	0.6076	95.841	1.982	3.062	3.057
Beethoven	1799	0.6170	967.118	2.985	2.709	2.706

Liszt	1849	0.6231	197.394	2.295	2.072	2.072
Skrjabin	1894	0.5766	24.135	1.383	1.596	1.600
Schoenberg	1913	0.6268	114.077	2.057	1.425	1.429
Stravinsky	1927	0.7556	7.269	0.861	1.308	1.313
Shostakovich	1940	0.6746	17.678	1.247	1.208	1.213
Ligeti	1965	0.6986	12.412	1.094	1.034	1.040

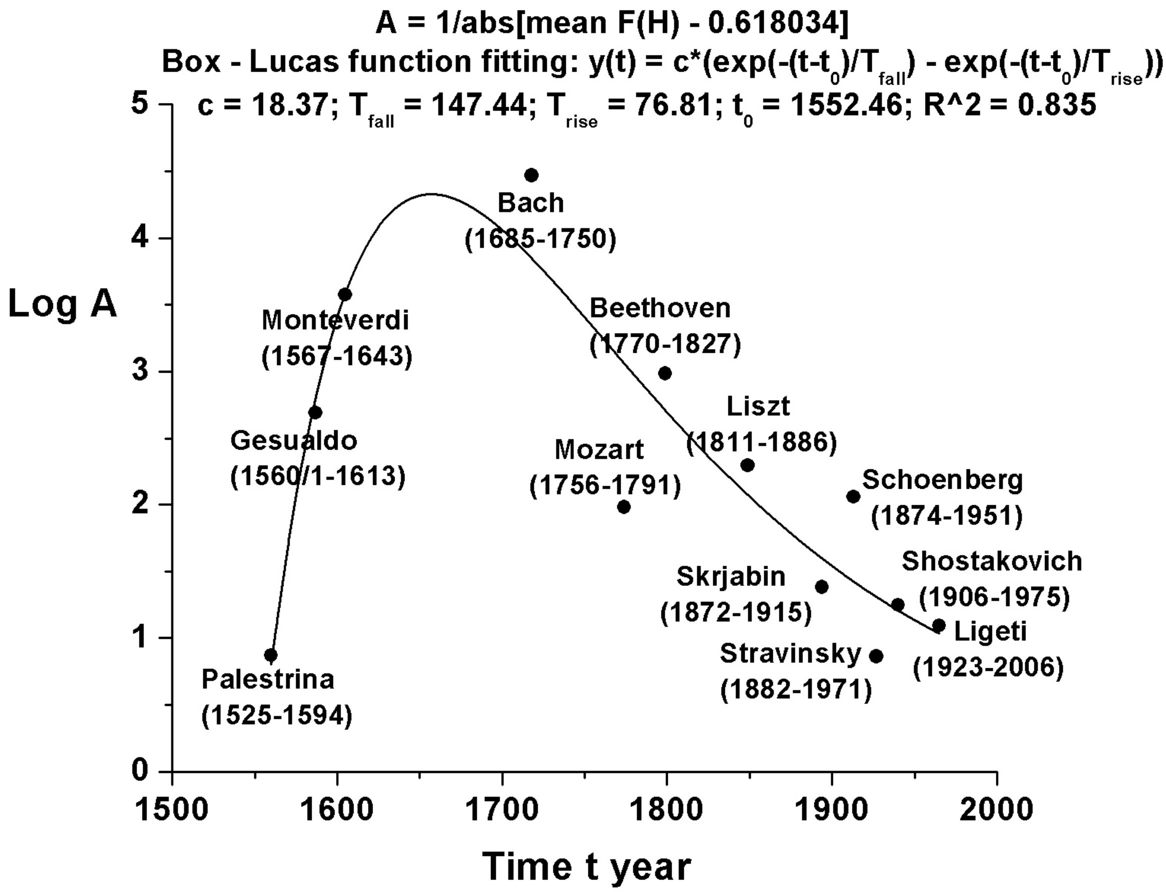

Figure 13. The musical echo of the Renaissance golden proportion
as revealed by the evolution of Log A (4 parameter Box-Lucas function fitting)

Another possibility is the use of the impulse function having three parameters and defined as

$$y(t) = c\exp\left(-\frac{t-t_0}{T}\right)\left[1-\exp\left(-\frac{t-t_0}{T}\right)\right]$$

yielding the results in Table 14 and Figure 14. The coincidence of both Box-Lucas and impulse function fitting is remarkable.

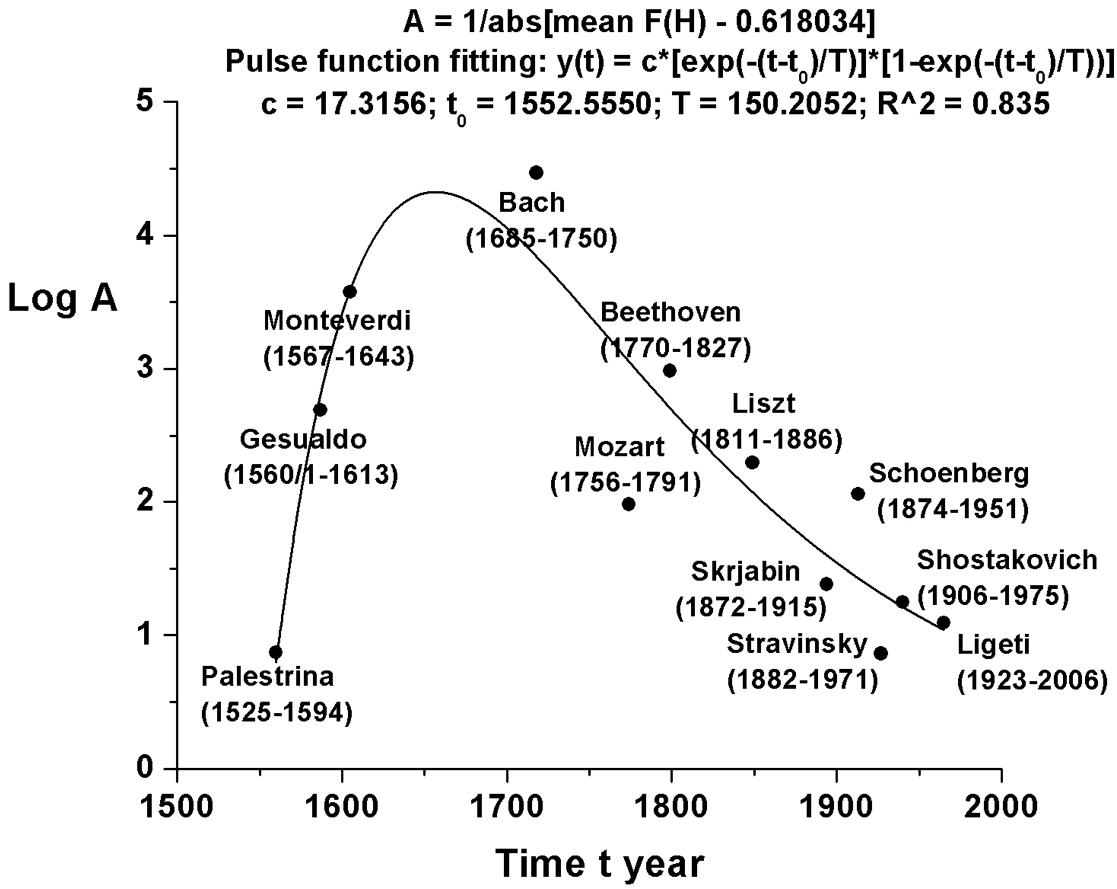

Figure 14. The musical echo of the Renaissance golden proportion
as revealed by the evolution of Log A (3 parameter impulse function fitting)

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Appendix

Table A1
H and *F*(*H*) for Palestrina

ID	Text	N	H	F(H)
Pls01	Ascendo 1. Motetto	1856	12	0.8475
Pls02	Ascendo 2. Kyrie	898	10	0.7728
Pls03	Ascendo 3. Gloria	1348	12	0.8435
Pls04	Ascendo 4. Credo	2120	9	0.7193
Pls05	Ascendo 5. Sanctus	595	9	0.7445
Pls06	Ascendo 5. Benedictus	563	8	0.7194
Pls07	Ascendo 7. Agnus Dei I	431	10	0.7610
Pls08	Ascendo 8. Agnus Dei II	487	12	0.8480
Pls09	Ave Regina Chant	137	3	0.7445

Pls10	Ave Regina Kyrie	687	11	0.8122
Pls11	Ave Regina Gloria	1357	8	0.6743
Pls12	Ave Regina Credo	2355	11	0.8191
Pls13	Ave Regina Sanctus	436	10	0.7729
Pls14	Ave Regina Benedictus	505	9	0.7525
Pls15	Ave Regina Agnus Dei I	396	7	0.5455
Pls16	Ave Regina Agnus Dei II	402	10	0.7886
Pls17	Missa Papae Kyrie	995	8	0.7035
Pls18	Missa Papae Gloria	1437	13	0.8984
Pls19	Missa Papae Credo	2385	9	0.7338
Pls20	Missa Papae Sanctus	1060	9	0.7481
Pls21	Missa Papae Benedictus	644	6	0.5994
Pls22	Missa Papae Agnus Dei I	711	10	0.7792
Pls23	Missa Papae Agnus Dei II	793	13	0.9067
Pls24	Missa Veni Kyrie	669	7	0.6099
Pls25	Missa Veni Gloria	1013	8	0.6614
Pls26	Missa Veni Credo	1596	10	0.7531
Pls27	Missa Veni Sanctus	722	11	0.8324
Pls28	Missa Veni Benedictus	576	9	0.7622
Pls29	Missa Veni Agnus Dei I	343	12	0.8630
Pls30	Missa Veni Agnus Dei II	415	7	0.5735
		$\overline{F(H)} = 0.7530 \pm 0.0893$		

Table A2
H and *F*(*H*) for Gesualdo

ID	Text	N	H	F(H)
Ges01	Belta, poi che te accendi	688	10	0.6221
Ges02	Deh, coprite il bel seno	591	9	0.6024
Ges03	Dolcissima mia vita	581	10	0.6145
Ges04	Itene, o miei sospiri	761	10	0.6491
Ges05	Moro, lasso, al mio duolo	671	11	0.6528
Ges06	O vos omnes	432	12	0.5833
Ges07	Merce grido piangendo	681	8	0.5918
		$\overline{F(H)} = 0.6166 \pm 0.0249$		

Table A3
H and *F*(*H*) for Monteverdi

ID	Text	N	H	F(H)
Mon01	Monteverdi - Dixit Dominus (Psalm 109)	3002	12	0,8028
Mon02	Monteverdi - Laudate pueri (Psalm 112)	1927	10	0,7286
Mon03	Monteverdi - Laetatus sum (Psalm 121)	2719	6	0,4777

Mon04	Monteverdi - Nisi Dominus (Psalm 126)	3138	6	0,5118
Mon05	Monteverdi - Lauda Jerusalem (Psalm 147)	2161	9	0,6858
Mon06	Monteverdi - Hymn: Ave maris stella	1411	7	0,5464
Mon07	Monteverdi - Magnificat	1240	7	0,5355
Mon08	Monteverdi - A un giro sol de'belli occhi	813	8	0,6335
Mon09	Monteverdi - Si, ch'io vorrei morire	886	9	0,6377
Mon10	Monteverdi - Vorrei baciarti, o Filli	2217	6	0,6229
		$\overline{F(H)} = 0.6183 \pm 0.0972$		

Table A4
H and *F*(*H*) for Bach

ID	Text	N	H	F(H)
Bach01	1. Prelude and Fugue No 1	1318	10	0,5948
Bach02	1. Prelude and Fugue No 2	1877	10	0,5685
Bach03	1. Prelude and Fugue No 3	2266	14	0,6827
Bach04	1. Prelude and Fugue No 4	2085	16	0,7108
Bach05	1. Prelude and Fugue No 5	1553	13	0,6542
Bach06	1. Prelude and Fugue No 6	1602	10	0,5449
Bach07	1. Prelude and Fugue No 7	2345	12	0,5970
Bach08	1. Prelude and Fugue No 8	2129	12	0,5867
Bach09	1. Prelude and Fugue No 9	1221	14	0,7322
Bach10	1. Prelude and Fugue No 10	2069	12	0,5988
Bach11	1. Prelude and Fugue No 11	1562	11	0,5583
Bach12	1. Prelude and Fugue No 12	1897	11	0,5651
Bach13	1. Prelude and Fugue No 13	1378	12	0,6277
Bach14	1. Prelude and Fugue No 14	1477	10	0,5423
Bach15	1. Prelude and Fugue No 15	2392	12	0,5560
Bach16	1. Prelude and Fugue No 16	1491	10	0,5265
Bach17	1. Prelude and Fugue No 17	1575	13	0,6832
Bach18	1. Prelude and Fugue No 18	1371	13	0,6207
Bach19	1. Prelude and Fugue No 19	1794	14	0,6711
Bach20	1. Prelude and Fugue No 20	3043	15	0,7026
Bach21	1. Prelude and Fugue No 21	1603	11	0,5958
Bach22	1. Prelude and Fugue No 22	1514	14	0,6955
Bach23	1. Prelude and Fugue No 23	1315	11	0,5932
Bach24	1. Prelude and Fugue No 24	2551	10	0,5076
Bach25	2. Prelude and Fugue No 1	1973	14	0,6984
Bach26	2. Prelude and Fugue No 2	1361	10	0,5871
Bach27	2. Prelude and Fugue No 3	1624	16	0,7956
Bach28	2. Prelude and Fugue No 4	2663	17	0,7570
Bach29	2. Prelude and Fugue No 5	2423	11	0,5761
Bach30	2. Prelude and Fugue No 6	1897	9	0,5071

Bach31	2. Prelude and Fugue No 7	1616	13	0,6714
Bach32	2. Prelude and Fugue No 8	1994	13	0,6153
Bach33	2. Prelude and Fugue No 9	1645	11	0,6170
Bach34	2. Prelude and Fugue No 10	2637	13	0,6435
Bach35	2. Prelude and Fugue No 11	2206	10	0,5254
Bach36	2. Prelude and Fugue No 12	1849	9	0,5203
Bach37	2. Prelude and Fugue No 13	2618	13	0,6429
Bach38	2. Prelude and Fugue No 14	2279	13	0,6441
Bach39	2. Prelude and Fugue No 15	2436	13	0,6831
Bach40	2. Prelude and Fugue No 16	2144	11	0,5896
Bach41	2. Prelude and Fugue No 17	2876	11	0,5741
Bach42	2. Prelude and Fugue No 18	4090	12	0,5689
Bach43	2. Prelude and Fugue No 19	1439	10	0,5587
Bach44	2. Prelude and Fugue No 20	2271	16	0,6319
Bach45	2. Prelude and Fugue No 21	4421	16	0,7356
Bach46	2. Prelude and Fugue No 22	2933	16	0,7092
Bach47	2. Prelude and Fugue No 23	2355	10	0,5176
Bach48	2. Prelude and Fugue No 24	1852	11	0,5767
		$\overline{F(H)} = 0.6180 \pm 0.0703$		

Table A5
H and *F*(*H*) for Mozart

ID	Text	N	H	F(H)
Moz01	Mozart D major K.284	10585	13	0,6357
Moz02	Mozart C major K.309	7577	10	0,5125
Moz03	Mozart A minor K.310	8117	15	0,653
Moz04	Mozart Bb major K.333	7496	12	0,6107
Moz05	Mozart A major K.331	9470	9	0,5583
Moz06	Mozart C minor K.457	6400	15	0,6570
Moz07	Mozart C major K.545	3628	12	0,6563
Moz08	Mozart D major K.311	7157	10	0,5391
Moz09	Mozart F major K.332	6868	14	0,6457
		$\overline{F(H)} = 0.6076 \pm 0.0530$		

Table A6
H and *F*(*H*) for Beethoven

ID	Text	N	H	F(H)
LvB01	LvB Sonata 1	7332	13	0,5573
LvB02	LvB Sonata 2	9340	24	0,7661
LvB03	LvB Sonata 3	11915	14	0,5446
LvB04	LvB Sonata 4	12248	18	0,6424

LvB05	LvB Sonata 5	7229	15	0,6159
LvB06	LvB Sonata 6	7171	17	0,5948
LvB07	LvB Sonata 7	9201	19	0,6172
LvB08	LvB Sonata 8	8396	18	0,6205
LvB09	LvB Sonata 9	5706	19	0,6746
LvB10	LvB Sonata 10	6623	14	0,6005
LvB11	LvB Sonata 11	10898	18	0,6822
LvB12	LvB Sonata 12	9497	16	0,6324
LvB13	LvB Sonata 13	8461	13	0,5426
LvB14	LvB Sonata 14	8597	12	0,5437
LvB15	LvB Sonata 15	11581	16	0,6198
LvB16	LvB Sonata 16	13439	19	0,6497
LvB17	LvB Sonata 17	7905	19	0,6405
LvB18	LvB Sonata 18	12428	13	0,5533
LvB19	LvB Sonata 19	3362	10	0,5580
LvB20	LvB Sonata 20	2937	15	0,7518
LvB21	LvB Sonata 21	14682	18	0,5752
LvB22	LvB Sonata 22	5802	18	0,6013
LvB23	LvB Sonata 23	15575	17	0,5526
LvB24	LvB Sonata 24	4619	18	0,6820
LvB25	LvB Sonata 25	5930	15	0,6260
LvB26	LvB Sonata 26	7416	17	0,6207
LvB27	LvB Sonata 27	6643	18	0,6294
LvB28	LvB Sonata 28	8467	15	0,5040
LvB29	LvB Sonata 29	21559	26	0,6232
LvB30	LvB Sonata 30	8713	19	0,6423
LvB31	LvB Sonata 31	8075	21	0,6537
LvB32	LvB Sonata 32	13468	23	0,6259
		$\overline{F(H)} = 0.6170 \pm 0.0570$		

Table A7

H and *F*(*H*) for Liszt

ID	Text	N	H	F(H)
Liszt01	Liszt - Concert Etude No.3 Un Sospiro	1495	19	0,6863
Liszt02	Liszt - Paganini Etude No.3 La Campanella	4278	17	0,6173
Liszt03	Liszt - Transzendental Etudes Eroica	3003	24	0,5744
Liszt04	Liszt - Transzendental Etudes Feux Follets	4420	23	0,6860
Liszt05	Liszt - Venezia e Napoli: 1. Gondoliera	2899	14	0,6609
Liszt06	Liszt - Venezia e Napoli: 2. Canzone	2211	13	0,6260
Liszt07	Liszt - Venezia e Napoli: 3. Tarantella	7731	14	0,4315
Liszt08	Liszt - Sonata h mol	15921	27	0,5892
Liszt09	Liszt - Hungarian Dance 1	2790	18	0,6441
Liszt10	Liszt - Hungarian Dance 5	1785	11	0,5322

Liszt11	Liszt - Hungarian Dance 6	3065	18	0,6803
Liszt12	Liszt - Hungarian Rhapsody	941	14	0,6865
Liszt13	Liszt - Liebestraume No. 3	1891	23	0,7002
Liszt14	Liszt - Valse Oubliee No.1	1861	16	0,6083
Liszt15	Liszt - Valse Oubliee No.2	4147	18	0,6294
		$\overline{F(H)} = 0.6231 \pm 0.0692$		

Table A8
H and *F*(*H*) for Skrjabin

ID	Text	N	H	F(H)
Skr01	Skrjabin Prelude op. 27 – No 1	355	10	0,4704
Skr02	Skrjabin Prelude op. 27 – No 2	222	9	0,6081
Skr03	Skrjabin Prelude op. 31 – 1	651	13	0,5453
Skr04	Skrjabin Prelude op. 31 – 4	155	9	0,5032
Skr05	Skrjabin Prelude op. 33 – 2	195	12	0,6308
Skr06	Skrjabin Prelude op. 33 – 3	212	9	0,5896
Skr07	Skrjabin Prelude op. 35 – 2	362	9	0,4586
Skr08	Skrjabin Prelude op. 37 – No 1	212	8	0,5189
Skr09	Skrjabin Prelude op. 37 – No 2	91	11	0,7363
Skr10	Skrjabin Prelude op. 48 – 2	224	10	0,4598
Skr11	Skrjabin Prelude op. 59	709	20	0,6897
Skr12	Skrjabin Prelude op. 67 – 1	338	9	0,5769
Skr13	Skrjabin Prelude op. 74 – 3	228	9	0,5921
Skr14	Skrjabin Piece op. 2, No 1	1150	16	0,7574
Skr15	Skrjabin Etude op. 8, No 4	747	9	0,5114
Skr16	Skrjabin Etude op. 8, No 5	1541	10	0,5120
Skr17	Skrjabin Etude op. 8, No 12	2301	11	0,5067
Skr18	Skrjabin Poem op. 32 – No 1	981	10	0,6575
Skr19	Skrjabin Počme tragique op.34	1001	11	0,6284
Skr20	Skrjabin Etude op. 42, No 4	787	10	0,5756
Skr21	Skrjabin Etude op. 42, No 5	3088	10	0,4828
Skr22	Skrjabin Sonate No 5, op. 53	7761	19	0,5588
Skr23	Skrjabin Sonate No 9, op. 68	4014	25	0,6682
Skr24	Skrjabin Poem op. 69 – No 2	539	11	0,6178
Skr25	Skrjabin Dance op. 73 – No 1 - Guirlandes	694	14	0,5130
Skr26	Skrjabin Dance op. 73 – No 2 – Flammes sombres	1051	13	0,6232
		$\overline{F(H)} = 0.5766 \pm 0.0813$		

Table A9
H and *F*(*H*) for Schoenberg

ID	Text	N	H	F(H)
Sch01	Verklaerte Nacht	15477	18	0.6144
Sch02	Mondestrunken	1197	16	0.6266
Sch03	Valse de Chopin	1146	16	0.6353
Sch04	Nacht (Passacaglia)	1108	23	0.6724
Sch05	Raub	661	14	0.6157
Sch06	Galgenlied	244	14	0.6116
Sch07	Die Kreuze	2042	15	0.6166
Sch08	Parodie	1329	20	0.6253
Sch09	O alter Duft	537	14	0.6089
Sch10	Piece for piano Op.33a	763	27	0.6619
Sch11	Six Little Piano Pieces Op.19	627	17	0.6061
		$\overline{F(H)} = 0.6268 \pm 0.0208$		

Table A10
H and *F*(*H*) for Stravinsky

ID	Text	N	H	F(H)
Str01	Adoration of the Earth	2490	19	0.7574
Str02	The Augurs of Spring	5139	12	0.6550
Str03	Ritual of Abduction	2794	16	0.6442
Str04	Spring Rounds	2805	34	0.8781
Str05	Ritual of the Rival Tribes	3267	36	0.8445
Str06	Procession of the Sage	738	23	0.6965
Str07	Dance of the Earth	1806	29	0.9147
Str08	The Sacrifice - Introduction	1994	23	0.7161
Str09	Mystic Circles	3085	15	0.6707
Str10	Glorification of the Chosen	1715	29	0.7767
Str11	Evocation of the Ancestors	1301	14	0.9101
Str12	Ritual Action of the Ancestors	2588	30	0.8876
Str13	Sacrificial Dance	5800	34	0.7445
Str14	The Firebird Suite (complete)	37659	28	0.7088
Str15	The Firebird Suite - Introduction	2919	36	0.9394
Str16	The Firebird's Dance	1015	19	0.9202
Str17	The Firebird Suite - Variations	3735	13	0.5971
Str18	The Princesses' Round Dance	1481	12	0.5692
Str19	The Infernal Dance	18912	22	0.6367
Str20	Berceuse	1877	21	0.7725
Str21	Finale	7733	23	0.7886
Str22	Symphony of Psalms 1	1878	24	0.7545

Str23	Symphony of Psalms 2	1494	20	0.6365
Str24	Symphony of Psalms 3	4214	27	0.714
		$\overline{F(H)} = 0.7556 \pm 0.1079$		

Table A11
H and *F*(*H*) for Shostakovich

ID	Text	N	H	F(H)
Sho01	Op.87 Prelude No.1 in C major	440	6	0.5545
Sho02	Op.87 Fugue No.1 in C major	172	6	0.7209
Sho03	Op.87 Prelude No.2 in A minor	323	8	0.6347
Sho04	Op.87 Fugue No.2 in A minor	247	10	0.6032
Sho05	Op.87 Prelude No.3 in G major	330	10	0.5606
Sho06	Op.87 Fugue No.3 in G major	407	9	0.7309
Sho07	Op.87 Prelude No.4 in E minor	429	7	0.5874
Sho08	Op.87 Fugue No.4 in E minor	453	6	0.6468
Sho09	Op.87 Prelude No.5 in D major	516	8	0.7267
Sho10	Op.87 Fugue No.5 in D major	312	7	0.6346
Sho11	Op.87 Prelude No.6 in B minor	321	14	0.6729
Sho12	Op.87 Fugue No.6 in B minor	367	13	0.6807
Sho13	Op.87 Prelude No.7 in A major	304	12	0.7928
Sho14	Op.87 Fugue No.7 in A major	483	15	0.8551
Sho16	Op.87 Fugue No.8 in F-sharp minor	390	13	0.7795
Sho17	Op.87 Prelude No.9 in E major	195	10	0.6821
Sho18	Op.87 Fugue No.9 in E major	573	8	0.6422
Sho19	Op.87 Prelude No.10 in C-sharp minor	430	20	0.6442
Sho20	Op.87 Fugue No.10 in C-sharp minor	404	7	0.5990
Sho21	Op.87 Prelude No.11 in B major	306	11	0.6830
Sho22	Op.87 Fugue No.11 in B major	611	8	0.6268
Sho23	Op.87 Prelude No.12 in G-sharp minor	476	11	0.7836
Sho24	Op.87 Fugue No.12 in G-sharp minor	480	7	0.5354
Sho25	Op.87 Prelude No.13 in F-sharp major	401	7	0.6509
Sho26	Op.87 Fugue No.13 in F-sharp major	250	8	0.7600
Sho27	Op.87 Prelude No.14 in E-flat minor	791	6	0.7155
Sho28	Op.87 Fugue No.14 in E-flat minor	394	6	0.5660
Sho29	Op.87 Prelude No.15 in D-flat major	1070	8	0.6654
Sho30	Op.87 Fugue No.15 in D-flat major	407	11	0.7101
Sho31	Op.87 Prelude No.16 in B-flat minor	354	7	0.6328
Sho32	Op.87 Fugue No.16 in B-flat minor	634	6	0.7319
Sho33	Op.87 Prelude No.17 in A-flat major	588	12	0.7823
Sho34	Op.87 Fugue No.17 in A-flat major	607	4	0.5634
Sho35	Op.87 Prelude No.18 in F minor	250	8	0.5840
Sho36	Op.87 Fugue No.18 in F minor	332	7	0.6145
Sho37	Op.87 Prelude No.19 in E-flat major	338	12	0.5740

Sho38	Op.87 Fugue No.19 in E-flat major	256	7	0.6367
Sho39	Op.87 Prelude No.20 in C minor	306	7	0.5523
Sho40	Op.87 Fugue No.20 in C minor	335	8	0.5910
Sho41	Op.87 Prelude No.21 in B-flat major	867	10	0.5686
Sho42	Op.87 Fugue No.21 in B-flat major	542	8	0.5923
Sho43	Op.87 Prelude No.22 in G minor	503	16	0.7435
Sho44	Op.87 Fugue No.22 in G minor	371	11	0.8032
Sho45	Op.87 Prelude No.23 in F major	378	10	0.7090
Sho46	Op.87 Fugue No.23 in F major	519	17	0.8266
Sho47	Op.87 Prelude No.24 in D minor	355	9	0.7042
Sho48	Op.87 Fugue No.24 in D minor	1015	10	0.5724
Sho49	Op.93 Symphony Nr.10 e-Moll - 1st Mov.	1056	11	0.8570
Sho50	Op.93 Symphony Nr.10 e-Moll - 2nd Mov.	790	10	0.7722
Sho51	Op.93 Symphony Nr.10 e-Moll - 3rd Mov.	259	9	0.8610
Sho52	Op.93 Symphony Nr.10 e-Moll - 4th Mov.	1194	11	0.6843
		$\overline{F(H)} = 0.6764 \pm 0.0886$		

Table A12
H and *F*(*H*) for Ligeti

ID	Text	N	H	F(H)
Lig01	Études pour piano 1 Désordre	3017	30	0,7676
Lig02	Étude 4: Fanfares	3142	26	0,6706
Lig03	Étude 5: Arc-en-ciel	3015	24	0,6577
		$\overline{F(H)} = 0.6986 \pm 0.0491$		