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RECONSTRUCTION OF EINSTEIN'S WORKS ON NOVEMBER 1915, REGARDING THE GRAVITATIONAL FIELD EQUATIONS AND THE PREDICTION OF THE GRAVITATIONAL TESTS

BY

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Abstract During the years 1912-1918, the creative efforts of Albert Einstein (b. 1879 – d. 1955) were directed towards the discovery of General Relativity Theory, name under which he meant a comprehensive theory of gravito-dynamic phenomena, including principia, mathematical equations, interconnections between space, time and matter and physical implications at all scales of matter aggregation. Out of this relatively large period of scientific activity, we focused our attention to a much more restrictive period, namely the week since 18-th to 25 November 1915, when the efforts of Einstein were for the first time successfully materialized in mathematical equations never infringed since that time on. The purpose of this paper is that to reconstitute the demonstrations left aside, for the sake of graphical space economy, in the two works published by Einstein on 18-th and 25-th November 1915 in Sitzungsberichte (Berlin). So, we hope to enlighten to a greater extent the line of reasoning which led to one of the outstanding discovery of XX-th century – the General Relativity Theory. At the same time, a historical explanation, concerning the priority of Einstein referred to other competitors, is given now, when a centennial celebration of another famous Einstein's discovery – the Special Theory of Relativity – does happen.

1. INTRODUCTION

Our purpose is to reconstitute the mathematical demonstrations of the two short communications, made by Albert Einstein on 18-th and 25-th November 1915, respectively, to the Prussian Academy of Sciences, by completing the missing details, in view of achieving a fluent and clear derivation of the expected results. In the first work, the field equations outside the (point-like) source, $R_{\mu\nu} = 0$, are used, in connection with a special gauge ($\sqrt{-g} = 1$), and in Cartesian coordinates, in view of deriving an approximate (one body) metric, able to predict all the physical tests proportional to $1/c^2$, c being the speed of light in empty space. It is proved that, owing to the special gauge assumed

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(which actually means that the elementary volume of the space-time manifold is not in any way changed by the presence of the gravitational field: $\sqrt{-g}cdtdxdydz = cdtdxdydz$), the first order approximation of the metric (that is the linear approximation in terms of the gravitational constant) is sufficient in itself for correctly evaluating the tests and no change is obtained by going over to the second order approximation (that is by performing calculations up to terms proportional to G^2). Thereafter, the metric is used in view of predicting the perihelion advance formula and the light deflection formula.

In the second work published by Einstein on 25-th November 1915, the field equations are given without demonstration. However, a justification of these equations can be done, by attentively pursuing the Einsteinian line of reasoning, namely, putting together the following requirements: 1) A linear relationship between three tensors, $R_{\mu\nu}$ - the curvature one, $T_{\mu\nu}$ - the matter one, and the combined tensor $g_{\mu\nu}T$ (T being the scalar of $T_{\mu\nu}$) is postulated based on the covariance principle. 2) The field equations $R_{\mu\nu} = -k\left(T_{\mu\nu} + ag_{\mu\nu}T\right)$, so obtained, must be submitted to two constraints a) the Newtonian law of universal attraction should be recovered in the classical limit, b) the Poisson equation should be recovered too in the same limit, 3) geodetic motion in the metric derived as a solution of the field equations is postulated as well.

First order approximation of the metrics

In this approximation the metric functions depend linearly on Newton's constant G and we have

$$(ds)^2 = g_{00}(r)(cdt)^2 + g_{jk}dx^j dx^k$$

$$g_{00} = 1 - K_0 \frac{GM}{c^2 r}, \qquad g_{jk} = -\left(\delta_{jk} + \frac{1}{c^2}a_{jk}\right), \tag{1}$$

$$a_{jk} = \frac{GM}{r}\left(K_1\delta_{jk} + K_2\frac{x_j x_k}{r^2}\right), \qquad (x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z),$$

Einstein determined the three constants ($K_0,\ K_1,\ K_2$) using the field equations away of a punctual body, placed in the origin of an inertial frame of reference. For this purpose, he adopted a Cartesian coordinate system and, possibly, in order to spare graphical space, gives the result without any further demonstration, namely

$$K_0 = 2,\ K_1 = 0,\ K_2 = 2, \tag{2}$$

hence his starting field equations are

$$R_{\mu\nu} = F_{\mu\nu} + S_{\mu\nu}$$

$$F_{\mu\nu} = -\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0; \qquad (3)$$

$$S_{\mu\nu} \equiv (\ln\sqrt{-g})_{,\mu\nu} - (\ln\sqrt{-g})_{,\alpha}\Gamma^{\alpha}_{\mu\nu} = 0$$

$$\sqrt{-g} = 1\,,\ \Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$$

Most probably the line of his not revealed demonstration has been performed along the following steps. He had first to fix the value of the K_0 constant from the condition to get Newton's law of universal attraction for the case of non-relativistic motion ($v<<c$), hence

$$K_0 = 2 \qquad (4)$$

Then he had to consider the linear approximation of the field equations (3) for the considered problem – the field of a punctual body placed at the origin of a reference frame

$$\Gamma^{\alpha}_{\mu\nu,\alpha} = 0\,,\ \sqrt{-g} = 1 \qquad (5)$$

In a detailed form the preceding equations become

$$\Gamma^{k}_{00,k} = -\frac{1}{2}K_0\frac{GM}{c^2}\Delta\left(\frac{1}{r}\right) \quad (=0\,,\ \text{for } r \neq 0) \qquad (6)$$

$$\Gamma^{l}_{jk} = \frac{1}{2}\frac{GM}{c^2}\left\{\begin{aligned} &K_1\left[\left(\frac{\delta^l_j}{r}\right)_{,k} + \left(\frac{\delta^l_k}{r}\right)_{,j} - \left(\frac{\delta_{jk}}{r}\right)^{,l}\right] \\ &+ K_2\left[\left(\frac{x_j x^l}{r^3}\right)_{,k} + \left(\frac{x_k x^l}{r^3}\right)_{,j} - \left(\frac{x_j x_k}{r^3}\right)^{,l}\right]\end{aligned}\right\}$$

$$\Gamma^{l}_{jk,l} = \frac{1}{2}\frac{GM}{c^2}\left\{\begin{aligned} &K_1\left[2\left(\frac{1}{r}\right)_{,jk} - \delta_{jk}\Delta\left(\frac{1}{r}\right)\right] \\ &+ K_2\left[\left(\frac{x_j x^l}{r^3}\right)_{,kl} + \left(\frac{x_k x^l}{r^3}\right)_{,jl} - \Delta\left(\frac{x_j x_k}{r^3}\right)\right]\end{aligned}\right\} = 0$$

After some computation we conclude that, for $r \neq 0$, the term proportional to K_2 from the expression of the quantity $\Gamma^l_{jk,l}$ vanishes and, consequently, the only possibility to satisfy the condition $\Gamma^l_{jk,l} = 0$ is given by

$$K_1 = 0, \tag{7}$$

The metrics now becomes

$$(dS)^2 = \left(1 - 2\frac{GM}{c^2 r}\right)(cdt)^2 - \left(\delta_{jk} + K_2 \frac{GM}{c^2 r^3} x_j x_k\right) dx^j dx^k , \tag{8}$$

and the remaining coefficient, K_2, can easily be determined from the condition of field gauge $\sqrt{-g} = 1$, namely

$$K_2 = 2 \tag{9}$$

We conclude with Einstein that, in the linear approximation (showing a linear dependence of the metric functions on Newton's G constant) the metrics is completely determined and given by

$$(dS)^2 = \left(1 - 2\frac{GM}{c^2 r}\right)(cdt)^2 - \left(\delta_{jk} + 2\frac{GM}{c^2 r^3} x_j x_k\right) dx^j dx^k \tag{10}$$

2. THE GRAVITATION FIELD EQUATIONS

A week after the great success with the exact prediction of the perihelion advance of the Mercury planet, i.e. on November 25, 1915, Einstein publishes the field equations of the general relativity again without any demonstration of their deduction. Taking into account the relatively short time elapsed between the two events (November 18, 1915 for the explanation of the perihelion advance of planets; and November 25, 1915 for the obtainment of the field equations of the gravitation), we believe that these achievements are tightly and necessarily correlated. In November 1915, Einstein concluded that between the matter tensor, $T_{\mu\nu}$, and the curvature tensor, $R_{\mu\nu}$, of a chronotropic universe with a given mass distribution, a linear relation should exist. In a general form, required by the covariance principle, this relationship has the form

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + \alpha g_{\mu\nu} T) \tag{11}$$

At the beginning of November, Einstein was still on the point to adopt the value zero for the coefficient α from the preceding equation. However, while considering the linear approximation of the metrics for the evaluation of the planet perihelion advance, he suddenly had an illumination („eine Erklärung"). Actually, he realized that that coefficient can be determined by analytical prolongation of metric functions inside the source, in the linear approximation and with the field gauge that earned him the success of November 18, 1915.

Let us return to the first of equations (6) and perform there the respective prolongation. We get

$$\Gamma^{k}_{00,k} = 2\pi K_0 \frac{GM}{c^2}\delta(\vec{r}) \tag{12}$$

In the given approximation and gauge, this result can be further transcribed as follows

$$R_{00} = -2\pi K_0 \frac{GM}{c^2}\delta(\vec{r}) \tag{13}$$

On the other side, with the help of equation (11), we can replace the quantity R_{00} from (13) with its equivalent expressed by merely physical quantities, namely

$$T_{00} + \alpha g_{00} T = \frac{1}{4} K_0 Mc^2 \delta(\vec{r}) \tag{14}$$

For $T_{\mu\nu} = Mc^2\delta(\vec{r})\delta_{0\mu}\delta_{0\nu}$, $g_{00} = 1$, it results $1+\alpha = \frac{1}{4}K_0$. Taking further into consideration the condition of smooth joining of Einstein's field equations to Newton's gravitation theory for the non- relativistic case, which delivers $K_0 = 2$, it results

$$\alpha = -\frac{1}{2} \tag{15}$$

The field equations get their final form given by Einstein (November 25, 1915)

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \tag{16}$$

These equations can be put in the equivalent form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{17}$$

thus evidencing the conservativity of the tensor $T_{\mu\nu}$ as a consequence of the fact that the covariant divergence of the tensor from the left side of the equality (17) (called sometimes Einstein's tensor) is identically equal to zero.

Second order approximation of the metrics

In order to consistently take into account the relativistic corrections proportional with $\frac{1}{c^2}$, a term in G^2 should be added in the temporal part of metric (10), so that we obtain

$$(dS)^2 = \left(1 - 2\frac{GM}{c^2 r} + 2K_3 \frac{G^2 M^2}{c^4 r^2}\right)(cdt)^2 - \left(\delta_{jk} + 2\frac{GM}{c^2 r^3} x_j x_k\right) dx^j dx^k \tag{18}$$

Einstein has done the computation of the coefficient K_3 with the help of the field equations that, in this case are

$$\Gamma^k_{00,k} = 2\Gamma^0_{0k}\Gamma^k_{00} \tag{19}$$

Also the direct determination of the connexion Γ^k_{00} has been performed using the general definition from (3) and the expression of the metric function g_{00} from (18). Consequently,

$$\Gamma^k_{00} = \frac{GM}{c^2 r^3} x^k - 2(K_3 + 1)\left(\frac{GM}{c^2 r^2}\right)^2 x^k \tag{20}$$

Also because $\left(\frac{x^k}{r^3}\right)_{,k} = 0$ and $\Gamma^0_{0k} \approx \delta_{jk}\Gamma^j_{00}$, the field equation (19) becomes

$$2(K_3 + 1)\left(\frac{GM}{c^2 r^2}\right)^2 = 2\Gamma^0_{0k}\Gamma^k_{00} = 2\delta_{jk}\Gamma^j_{00}\Gamma^k_{00} = 2\left(\frac{GM}{c^2 r^2}\right)^2 \tag{21}$$

from where we have

$$K_3 = 0 \tag{22}$$

In other words, the metric (10), in the linear approximation of the metric functions (by series development on G constant powers) gives correctly the relativistic effects proportional to $1/c^2$ (because the special gauge form $\sqrt{-g} = 1$). For the computation of the perihelion advance of planets it is, therefore, sufficient to adopt the metric (1) and the variational principle of geodesic motion.

$$\frac{d^2 x^\alpha}{dS^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS} = 0 \tag{23}$$

The relativistic equation of motion

Detailing equation (23), we get two equations, namely

$$\frac{d^2x^j}{dS^2}+\Gamma^j_{00}\left(\frac{dx^0}{dS}\right)^2+\Gamma^j_{kl}\frac{dx^k}{dS}\frac{dx^l}{dS}=0$$
$$\frac{d^2x^0}{dS^2}+2\Gamma^0_{0j}\frac{dx^0}{dS}\frac{dx^j}{dS}=0 \tag{24}$$

However, by the very definition of afine connections, it results that the exact form of Christoffel symbol Γ^0_{0j} is

$$\Gamma^0_{0j}=(\ln\sqrt{g_{00}})_{,j} \tag{25}$$

so that the second equation (24) can be restricted to the form

$$\frac{d}{dS}\left(g_{00}\frac{dx^0}{dS}\right)=0 \tag{26}$$

At the same time, we have to take into consideration the Lagrange function equation of motion (which is a constant equal to 1)

$$L=g_{00}\left(\frac{dx^0}{dS}\right)^2-\left(\delta_{jk}+\frac{1}{c^2}a_{jk}\right)\frac{dx^j}{dS}\frac{dx^k}{dS}\equiv 1 \tag{27}$$

It is convenient now to introduce the invariant time τ (of the co-moving observer), as preferred by Einstein, and to write

$$\frac{dx^l}{dS}=\frac{1}{c}\frac{dx^l}{d\tau}=\frac{1}{c}v^l,\ \delta_{jk}\frac{dx^j}{dS}\frac{dx^k}{dS}=\frac{1}{c^2}\left(\frac{d\vec{r}}{d\tau}\right)^2=\frac{\vec{v}^{\,2}}{c^2},$$
$$g_{00}\frac{dx^0}{dS}\approx\sqrt{g_{00}\left(1+\vec{v}^{\,2}/c^2\right)} \tag{28}$$

Because g_{00} is given in (10) under the form $g_{00}=1-2\frac{GM}{c^2r}$, it results for $g_{00}\frac{dx^0}{dS}$ the expression

$$g_{00}\frac{dx^0}{dS}\approx\left[\left(1-2\frac{GM}{c^2r}\right)\left(1+\vec{v}^{\,2}/c^2\right)\right]^{1/2}\approx 1+\frac{1}{c^2}\left(\frac{1}{2}\vec{v}^{\,2}-\frac{GM}{r}\right)=$$

$$=1+\frac{1}{mc^2}\left(\frac{1}{2}m\vec{v}^{\,2}-G\frac{mM}{r}\right)=1+\frac{\varepsilon_N}{mc^2}\qquad(29)$$

where ε_N is the planet (mechanical) energy in Newtonian normalization.

The first equation of (24) can be written in the form

$$\frac{d^2x^j}{dS^2}+c^2\Gamma^j_{00}\frac{f^2(\varepsilon_N)}{g^2_{00}}+\Gamma^j_{kl}v^kv^l=0\,.\qquad(30)$$

Here $f(\varepsilon_N)$ is the integration constant from (26), whose its explicit form is given in (29),

namely $f(\varepsilon_N)=1+\dfrac{\varepsilon_N}{mc^2}$. At the same time $c^2\Gamma^j_{00}=\dfrac{GM}{r^3}\left(1-2\dfrac{GM}{c^2r}\right)x^j$ can be obtained from (20) with the specification (22) about the metric constant K_3, g_{00} results from (20) with the same specification (22), thus $g_{00}=1-2\dfrac{GM}{c^2r}$,

and $\Gamma^j_{kl}=\dfrac{GM}{c^2r^3}\left(2\delta_{kl}-3\dfrac{x_kx_l}{r^2}\right)x^j$ results from (6) with the specifications (7) and (9). Putting to use all these details and performing the needed computations, equation (30) becomes

$$\frac{d^2\vec{r}}{d\tau^2}+\frac{GM}{r^3}\left\{1+\left(f^2-1\right)+2\frac{GM}{c^2r}+2\vec{v}^{\,2}/c^2-\frac{3}{c^2r^2}\left(\vec{v}\,\vec{r}\right)^2\right\}\vec{r}=0\qquad(31)$$

where $\vec{v}=\dfrac{d\vec{r}}{d\tau}$ and $d\tau=\dfrac{1}{c}dS$.

Einstein approximates the motion equation (31) in two stages and the results of both are given in the same paper on November 18, 1915, published at Berlin in a “Sitzungsberichte”. In the first stage, some day in the summer of 1915, he admitted the approximation $f=1$ and obtained the equation

$$\frac{d^2\vec{r}}{d\tau^2}+\frac{GM}{r^3}\left\{1+2\frac{GM}{c^2r}+2\frac{\vec{v}^{\,2}}{c^2}-\frac{3}{c^2r^2}\left(\vec{v}\,\vec{r}\right)^2\right\}\vec{r}=0\,,\qquad(32)$$

with the help of which the exact values for the planetary perihelion advance have been predicted by

$$\delta\varphi = 6\pi \frac{GM}{c^2 a\left(1-e^2\right)} \tag{33}$$

as well as for the deviation of the light trajectory by the passage near the solar disc, namely

$$\delta\psi = 4\frac{GM}{c^2 R} \tag{34}$$

It is a matter of fact that Einstein did not published the equations (32) - (34), earlier than on 18-th November, yet, he possibly communicated these results to some confident friends, in order to find out their reaction and, perhaps, for priority purposes either. At the beginning of the fall of 1915, Einstein had the unpleasant surprise to discover an error in his demonstration, namely that the value of the quantity f is not 1 but rather

$$f = 1 + \frac{1}{c^2}\left(\frac{1}{2}\vec{v}^{\,2} - \frac{GM}{r}\right) \tag{35}$$

Thus, the quantity ($f^2 - 1$) in (31) is

$$\left(f^2 - 1\right) = \left(\vec{v}^{\,2}/c^2 - 2\frac{GM}{c^2 r}\right) \tag{36}$$

and accordingly, it does not vanish. From (31) and (36) he concludes that the relationship (32) does not hold true and that the correct equation, in agreement with the variational principle of the minimal action, is

$$\frac{d^2\vec{r}}{d\tau^2} + \frac{GM}{r^3}\left\{1 + \frac{3}{c^2 r^2}\left(\vec{v}\,x\,\vec{r}\right)^2\right\}\vec{r} = 0 \tag{37}$$

At the same time, he brought forward the following ingenious solution to correct equation (32) and make it formally coincident with equation (37). Thus, he writes the equation (32) in the equivalent form

$$\frac{d^2\vec{r}}{d\tau^2} + \frac{GM}{r^3}\left\{\left[1 - \frac{2}{c^2}\left(\frac{1}{2}\vec{v}^{\,2} - \frac{GM}{r}\right)\right] + \frac{3}{c^2 r^2}\left(\vec{v}\,x\,\vec{r}\right)^2\right\}\vec{r} = 0 \tag{38}$$

then performs a time exchange by

$$\tau\left[1 - \frac{2}{c^2}\left(\frac{1}{2}\vec{v}^{\,2} - \frac{GM}{r}\right)\right]^{1/2} = \tau_E\,,\ \frac{d\,\vec{r}}{d\,\tau_E} = \vec{v}_E \tag{39}$$

The reason resides in the fact that, in a first relativistic approximation the ratio τ / τ_{E} represents a prime integral of the motion and brings the motion equation to the form (37), with the sole difference that instead of τ and $\vec{v}$ we have now τ_{E} and $\vec{v}_{E}$. Finally, Einstein drops out the index E as being useless. In this way, Einstein demonstrates that the results (33) and (34) (obtained by him in the summer of 1915) rest valid, in spite of the fact that these have been obtained with the help of the equation of motion (32), which violates the variational principle of minimal action. The equations (33) and (34) represent relativistic corrections of specific geometric trajectories (a rosette and a hyperbola, respectively) where the time does not intervene any more (it was eliminated between the prime integral of the energy and the prime integral of the angular momentum). Therefore, the same result one gets by using either a co-moving time τ, or a time $k\tau$, where k is a prime integral of the motion. Historically it is to be mentioned that Jean Chazy gives, in his treatise "La Mécanique Celeste et la Theorie de la Relativité" (1928) the formula (32) under the name of "Einstein's formula" and not the exact formula (37). Had Chazy would find out the equation of motion from Einstein's memoir dated on November 18, 1915, and not earlier, he would had no reason not citing in his book the exact formula (37). Clearly more research on Einstein's correspondence with his friends in the summer of 1915 would succeed in dating equation (32), which, surely, precedes equation (37).

The motion integral and Binet's equation

The approximate integration of equation (37) gives the prime integrals of the motion

$$A = \frac{1}{2}\vec{u}^{2} + \Phi, \; B^{2} = \left(\vec{u} \times \vec{r}\right)^{2} = \left(r^{2}\frac{d\varphi}{dS}\right)^{2}, \tag{40}$$

where

$$\vec{u} = \frac{\vec{v}}{c}, \; \Phi = -\frac{1}{2}\frac{\alpha}{r}\left(1 + \frac{B^{2}}{r^{2}}\right), \; \alpha = 2\frac{GM}{c^{2}}$$

and the quantities A and B have been computed in the Newtonian approximation, resorting to Kepler's laws and denoting by *a* the great semi-axis of the planetary orbit,

$$A \approx -\frac{1}{4}\frac{\alpha}{a}\,,\ B^2 \approx \frac{1}{2}\alpha a\left(1-a^2\right)$$

It remains now to transform the quantities Φ and $\vec{u}^{\,2}$. One gets immediately

$$\Phi = -\frac{1}{2}\left(\frac{\alpha}{a}\right)X - \frac{1}{4}\left(\frac{\alpha}{a}\right)^2\left(1-e^2\right)X^3\,,\ X \equiv \frac{a}{r}\,,$$

then, locating the motion in the elliptical plane, we find

$$\vec{u}^{\,2} = \frac{(dr)^2 + r^2(d\varphi)^2}{(dS)^2} = \left(\frac{dr}{dS}\right)^2 + \frac{B^2}{r^2} = \frac{B^2}{r^2}\left[\left(\frac{dX}{d\varphi}\right)^2 + X^2\right]$$
$$= \frac{1}{2}\left(\frac{\alpha}{a}\right)\left(1-e^2\right)\left[\left(\frac{dX}{d\varphi}\right)^2 + X^2\right]$$

(in order to obtain this result we took into account that $\frac{dr}{dS} = -\frac{B}{a}\frac{dX}{d\varphi}$)

The energy equation now becomes

$$-\frac{1}{4}\frac{\alpha}{a} = \frac{1}{4}\frac{\alpha}{a}\left(1-e^2\right)\left[\left(\frac{dX}{d\varphi}\right)^2 + X^2\right] - \frac{1}{2}\frac{\alpha}{a}X - \frac{1}{4}\left(\frac{\alpha}{a}\right)^2\left(1-e^2\right)X^3$$

After performing the necessary simplifications, this equation gets the form

$$\left(\frac{dX}{d\varphi}\right)^2 = P_3(X),\ P_3(X) \equiv \varepsilon X^3 - X^2 + \frac{2}{1-e^2}X - \frac{1}{1-e^2} \tag{41}$$

where $\varepsilon = \frac{\alpha}{a}$ represents a small perturbation, responsible for the displacement of the planetary perihelion. The preceding equation has the integrated form of the relativistic Binet equation. The standard form results from (41) by differentiation and then by returning to the variable r

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) = \frac{1}{p} + 3\mu\left(\frac{1}{r}\right)^2,\ p = a\left(1-e^2\right),\ \mu = \frac{GM}{c^2} \tag{42}$$

The computation of the planetary perihelion advance

Einstein preferred to work with equation (41) and to evaluate the integral

$$\delta\varphi = 2\int_{X_1}^{X_2} \frac{dX}{\sqrt{P_3(X)}} \tag{43}$$

where $X_1 = \frac{1}{1+e} + O(\varepsilon)$ and $X_2 = \frac{1}{1-e} + O(\varepsilon)$ are the roots of $P_3(X)$ close to the classical turning points. The movement occurs between these points $X_1 < X < X_2$, and the third root, determinated by the perturbation ε, is much more remote then the former two, $X_3 >> X_1$, $X_3 >> X_2$, inasmuch as its effect on the integral (43) can be treated as a perturbation.. Consequently, $X_3 = \frac{1}{\varepsilon} - \frac{2}{1-e^2} + O(\varepsilon)$ and we can write

$$P_3(X) \equiv \varepsilon X^3 - X^2 + \frac{2}{1-e^2}X - \frac{1}{1-e^2};$$

$$\sqrt{P_3(X)} \approx \left[1 - \frac{1}{2}\varepsilon(X_1 + X_2)\right]\frac{1}{1+\frac{1}{2}\varepsilon X}\sqrt{(X_2 - X)(X - X_1)};$$

$$\delta\varphi = 2\left[1 + \frac{1}{2}\varepsilon(X_1 + X_2)\right]\int_{X_1}^{X_2} \frac{\left[1 + \frac{1}{2}\varepsilon X\right]dX}{\sqrt{(X_2 - X)(X - X_1)}}$$

$$= 2\pi\left[1 + \frac{1}{2}\varepsilon(X_1 + X_2)\right]\left[1 + \frac{1}{4}\varepsilon(X_1 + X_2)\right]$$

$$\approx 2\pi\left[1 + \frac{3}{4}\varepsilon(X_1 + X_2)\right]$$

The advance of the planetary perihelion is thus

$$\delta\varphi - 2\pi = \frac{3}{2}\pi\varepsilon(X_1 + X_2) \approx \frac{3}{2}\pi\left(2\frac{GM}{c^2 a}\right)\frac{2}{1-e^2} = 6\pi\frac{GM}{c^2 a(1-e^2)} \tag{44}$$

The next day (on November 19, 1915) David Hilbert congratulated Albert Einstein for his success regarding the advance of the Mercury planet perihelion by sending him a friendly telegram.

Einstein gave no details about his computation of the deviation of the light trajectory in the gravitational field of the Sun. We suppose, however, that he followed a standard method, namely starting from the equation (41) of the perturbed ellipse, written in a convenient form, evidencing the parameters $p = a(1-e^2)$ and $\mu = \frac{GM}{c^2}$, namely

$$\left[\frac{d}{d\varphi}\left(\frac{1}{r}\right)\right]^2+\left(\frac{1}{r}\right)^2=\frac{2}{p}\left(\frac{1}{r}\right)-\frac{1}{ap}+2\frac{\mu}{r^3}$$

and then passing to the limit $p \to \infty$ in view of going over from the ellipse to the hyperbola

$$\left[\frac{d}{d\varphi}\left(\frac{1}{r}\right)\right]^2+\left(\frac{1}{r}\right)^2=2\frac{\mu}{r^3} \tag{45}$$

Finally, one evaluates the angle between the asymptotes of the hyperbola, and this is precisely the expected effect.

Einstein's forerunners

The problem of the advance of the perihelion of the Mercury planet has been stated since the middle of the XIX-th century, when Joseph Leverrier set forth the hypothesis that the perturbations observed in the movement of this planet, not explained by Newton's gravitation theory, could be due to a planet not yet discovered, called by him Vulcan. In 1874, F.Tisserand, starting from the hypothesis that Newton's force depends also on the planet speed and well inspired by the analogy between gravitation and electrodynamics, succeeds to obtain formula (33), excepting the coefficient 6. In 1893, Oliver Heaviside proposed the use of Maxwell equations in gravitation with the following amendments: (i) there exist only positive gravitational charges $q_G = m\sqrt{G} > 0$ and (ii) the gravitational charges always attract each other with the force $f_G = -q_{1G}q_{2G}/r^2$, but not even he was able to explain completely the enigma of the planet Mercury. In the meantime, the idea that the gravity is an interaction propagating through the space with the light velocity was brought into credit by the contributions of Henri Poincaré (1906), Gunnar Nordsröm (1912), and mainly by the paper of Paul Gerber (1898). The latter started from the gravity propagation hypothesis and arrived at the following variational principle

$$\delta\int L\left(\vec{r},\vec{v}\right)dt=0\,,\ L=\frac{1}{2}\vec{v}^{\,2}+G\frac{mM}{r}\left[1+\frac{3}{c^2r^2}\left(\vec{v}\,\vec{r}\right)^2\right],$$

from where he deduced the complete formula (33) of the advance of the planetary perihelion. Paul Gerber, however, remains as a mere lucky forerunner because his theory neither can be accommodated with the restricted relativity, nor can be expressed as a field theory.

A problem of much concern of Einstein was to give a convincing answer to the question why the relativistic description of the gravitational field needs 10 potentials and not less (for instance 4, as in electrodynamics, or even 1) ? He tried to reduce the problem ad absurdum if there were enough a single potential, imaging a „gedanken Experiment“ with a free-falling box filled with radiation, but Max von Laue called naive this argument of Einstein. A subsequent discussion challenged by R. H. Dicke led to the conclusion that the scalar gravity satisfies the equivalence principle. In 2003, one of us (N.I-P) builds a scalar model of the gravity, which renders correctly the 4 tests.

3. CONCLUSIONS

The two papers of Albert Einstein dated on 18 and 25 November 1915 highlight a key moment for the thoughts that eventually led to the discovery of the General Relativity as a viable theory of the dynamic gravitational phenomena. Published in a lapidary form, apparently sparing graphical space, the two articles, the first with the three relativistic tests of the gravity and the second establishing the field equations, appeared in a crucial moment of crisis, when the efforts to create the gravito-dynamics based on the restricted relativity failed and the only chance was Einstein's „General Relativity“. In 1914 Einstein did not know yet how he will couple the gravitational field to its sources, but in the late fall of 1915 it occurred the necessary clarification. The equivalence principle works in the space free of sources, where the curvature scalar is zero, together with the scalar of the matter tensor. The solutions of the field in this domain should be prolonged up to the source zone where the two scalars differ from zero inasmuch as to fulfill the following purposes: (i) to result a linear relation between the tensors $R_{\mu\nu}$, $T_{\mu\nu}$ and $g_{\mu\nu}T$ or (equivalently) between the tensors $T_{\mu\nu}$, $R_{\mu\nu}$ and $g_{\mu\nu}R$ and (ii) to recover, in the non-relativistic case, the Poisson equation. In the epoch of working out the general relativity, Einstein was so much concerned with the equality of various types of motion inasmuch as he was conducted to contest the role of the inertia principle (and the corresponding Poincaré transformation group) in the enlightenment of physical theories and avoided to use the inertial frame of the observer. But precisely in this reference frame it is possible to define the mechanical Lagrangean $L = -m_0 c dS / dt$, as well as the concept of „gravitational refractive index“, standing at the ground of the discovery of the 4-th test of the general relativity theory by Irving Shapiro in 1961, namely „the retardation of a radar signal caused by passing through an intense gravitational field“

However, we have to point out the fact that Einstein's attitude regarding the inertia principle changed since 1938 when, working with Hoffmann and Infeld in the many body gravitational problem, he has been obliged to admit that the mass center moves inertially. In the controversy about the existence of black holes with finite spatial extension, Einstein adopted, likewise Schwarzschild, a negative position (that is that there exist only punctual black holes surrounded by a horizon with constant area $S = 16\pi\mu^2$, as a result of a strong deviation of the geometric manyfold from the Euclidean geometry at inter-particle distances of the order of magnitude $\mu = GM / c^2$). In the whole Einsteinean thinking one may remark a priority of the intuition based on observation and experiment, in contrast to pure mathematical speculations, which should be subjected to a lucid control. We believe that the reconstruction of the two papers dated on November 1915, with details, thus allowing to young scientists to follow Einstein's deep physical thinking line, is most useful and instructive.

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